

Deformation Expression for Elements of Algebras (IV)

–Matrix elements and related integrals–

Hideki Omori*
Tokyo University of Science

Yoshiaki Maeda†
Keio University

Naoya Miyazaki‡
Keio University

Akira Yoshioka §
Tokyo University of Science

September 2, 2011

Contents

1	Fundamental facts for star-products	2
1.1	Generic properties of *-exponential functions of quadratic forms	5
1.2	Periodicity and the exchanging interval	5
1.2.1	Rules of setting slits and evaluations	6
2	Elements defined by integrals	7
2.1	Vacuums and pseudo-vacuums	8
2.1.1	The pseudo-vacuum	10
2.1.2	No other idempotent element	11
2.2	Two inverses in generic ordered expressions	12
2.2.1	Remarks on ordinary calculus	15
2.3	Integrals along closed paths	16
2.3.1	Residues and secondary residues	18
2.3.2	Residues and Laurent coefficients	21
3	The differential equation $\frac{\nabla}{d\sigma} f_\sigma(u, v) = :(\frac{1}{i\hbar} u \circ v + \alpha) :_K * _K f_\sigma(u, v)$	23
3.1	Co-moving expression parameters	26

*Department of Mathematics, Faculty of Sciences and Technology, Tokyo University of Science, 2641, Noda, Chiba, 278-8510, Japan, email: omori@ma.noda.tus.ac.jp

†Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1, Hiyoshi, Yokohama, 223-8522, Japan, email: maeda@math.keio.ac.jp

‡Department of Mathematics, Faculty of Economics, Keio University, 4-1-1, Hiyoshi, Yokohama, 223-8521, Japan, email: miyazaki@hc.cc.keio.ac.jp

§Department of Mathematics, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Tokyo, 102-8601, Japan, email: yoshioka@rs.kagu.tus.ac.jp

4	Matrix elements in generic ordered expressions	30
4.1	Matrix representations	33
4.1.1	Representations by Laurent expansions	35
4.1.2	Fourier series of alternating 2π -periodic functions	36
4.2	Diagonal matrix expressions and applications	37
4.2.1	Analytic continuation of inverses	39

Keywords: Weyl algebra, Polar elements, Vacuum, Pseudo-vacuum, Continuation of inverses

Mathematics Subject Classification(2000): Primary 53D55, Secondary 53D17, 53D10

In this note, we mainly consider the extended Weyl algebra of two generators (u, v) , that is, the algebra generated by u, v with the fundamental commutation relation $u*v - v*u = -i\hbar$. Weyl algebra is realized on the space $\mathbb{C}[u, v]$ by defining various product $*_K$ depending on a symmetric matrix K called the expression parameter. Via such expressions and ordinary calculus one can treat various transcendental elements such as $*$ -exponential functions and elements obtained by integrations.

In particular, setting $\frac{1}{i\hbar}u \circ v = \frac{1}{2}(u*v + v*u)$, we show that the $*$ -exponential function $e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha)}$ has singular points depending on the expression parameters. The important view point in this chapter is that the expression parameter K is moving by the “individual time parameter” τ .

In such an extended Weyl algebra, there are three idempotent elements, called vacuum, bar-vacuum and pseudo-vacuum, each of them give a matrix representation of Weyl algebra.

Furthermore the $*$ -exponential function $e_*^{t\frac{1}{i\hbar}u \circ v}$ is rapidly decreasing on the imaginary axis. This defines two different $*$ -inverses of $\frac{1}{i\hbar}u \circ v$. By using this, the analytic continuation of $(\alpha + \frac{1}{i\hbar}u \circ v)^{-1}$ are defined.

Note that the numbers α such that $(\alpha + \frac{1}{i\hbar}u \circ v)^{-1}$ does not exist is called the “spectre”. It is remarkable that the spectra depends on the expression parameters. Several interesting properties of elements defined by integrations will be given.

1 Fundamental facts for star-products

For an arbitrary fixed 2×2 -complex symmetric matrix K , we set $\Lambda = K + J$ where J is the standard skew-symmetric matrix $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. We define a product $*_\Lambda$ on the space of polynomials $\mathbb{C}[u_1, u_2]$ by the formula

$$(1.1) \quad f *_\Lambda g = f e^{\frac{i\hbar}{2}(\sum \overleftarrow{\partial}_{u_i} \Lambda^{ij} \overrightarrow{\partial}_{u_j})} g = \sum_k \frac{(i\hbar)^k}{k! 2^k} \Lambda^{i_1 j_1} \dots \Lambda^{i_k j_k} \partial_{u_{i_1}} \dots \partial_{u_{i_k}} f \partial_{u_{j_1}} \dots \partial_{u_{j_k}} g.$$

It is known and not hard to prove that $(\mathbb{C}[u_1, u_2], *_\Lambda)$ are mutually isomorphic associative algebras for every K . The isomorphism class is called the Weyl algebra, denoted by $(W_2; *)$. If K is fixed, then every element $A \in (W_2; *)$ is expressed in the form of ordinary polynomial, which we denote by $:A:_K \in \mathbb{C}[u_1, u_2]$. For instance

$$:u_1 * u_1:_K = (u_1)^2 + \frac{i\hbar}{2} K_{11}, \quad :u_1 * u_2:_K = u_1 u_2 + \frac{i\hbar}{2} (K_{12} - 1), \quad :u_2 * u_1:_K = u_1 u_2 + \frac{i\hbar}{2} (K_{12} + 1).$$

Note that $*_K$ -product formula gives a way of univalent expression for elements of $(W_2; *)$. Via univalent expressions, one can consider topological completion of the algebra, and transcendental elements.

By the formulation of orderings, the intertwiner between K -ordered expression and K' -ordered expression is explicitly given as follows:

Proposition 1.1 *For every $K, K' \in \mathfrak{S}(2)$, the intertwiner is defined by*

$$(1.2) \quad I_K^{K'}(f) = \exp\left(\frac{i\hbar}{4} \sum_{i,j} (K'^{ij} - K^{ij}) \partial_{u_i} \partial_{u_j}\right) f \quad (= I_0^{K'} (I_0^K)^{-1}(f)),$$

which gives an isomorphism $I_K^{K'} : (\mathbb{C}[\mathbf{u}]; *_{K+J}) \rightarrow (\mathbb{C}[\mathbf{u}]; *_{K'+J})$. Namely, for any $f, g \in \mathbb{C}[\mathbf{u}]$, we have

$$(1.3) \quad I_K^{K'}(f *_{\Lambda} g) = I_K^{K'}(f) *_{\Lambda'} I_K^{K'}(g),$$

where $\Lambda = K+J$, $\Lambda' = K'+J$.

Intertwiners do not change the algebraic structure $*$, but these change the expression of elements by the ordinary commutative structure.

In what follows, we use the notation $*_K$ instead of $*_{\Lambda}$, since the skew-part J is fixed on the standard skew-matrix. As in the case of one variable, infinitesimal intertwiner

$$dI_K(K') = \frac{d}{dt} \Big|_{t=0} I_K^{K+tK'} = \frac{i\hbar}{4} K'_{ij} \partial_{u_i} \partial_{u_j}$$

is viewed as a flat connection on the trivial bundle $\coprod_{K \in \mathfrak{S}(2)} \text{Hol}(\mathbb{C}^2)$. The equation of parallel translation along a curve $K(t)$ is given by

$$(1.4) \quad \frac{d}{dt} f_t = dI_K(\dot{K}(t)) f_t, \quad \dot{K}(t) = \frac{d}{dt} K(t),$$

but this may not have a solution for some initial function.

Associated with $H_* \in (W_2; *)$, the $*$ -exponential function of H_* is defined by the evolution equation

$$(1.5) \quad \frac{d}{dt} f_t = :H_* :_K * f_t, \quad f_0 = 1.$$

If the real analytic solution of (1.5) exists then the solution is denoted by $:e_*^{tH_*} :_K$.

In what follows we set (u_1, u_2) by (u, v) . Note that a quadratic forms with discriminant 0 is essentially u^2 by a linear change of generators, and $*$ -exponential $e_*^{tu^2}$ is treated in [9]. Note also that $2u \circ v = u * v + v * u$ is a representative of quadratic forms with discriminant 1 by a linear change of generators. We take our attention to the $*$ -exponential function of a quadratic form $2u \circ v$ under a general expression parameter $K = \begin{bmatrix} \delta & c \\ c & \delta' \end{bmatrix}$. By noting that $: \frac{1}{i\hbar} u \circ v :_K = \frac{1}{i\hbar} uv + \frac{1}{2}c$, the equation (1.5) for $H_* = \frac{2}{i\hbar} u \circ v$ is written precisely as

$$(1.6) \quad \begin{aligned} \frac{d}{dt} f_t(u, v) = & \left(\frac{c}{2} + \frac{1}{i\hbar} uv \right) f_t(u, v) + ((c+1)u + \delta v) \partial_u f_t(u, v) + (\delta' u + (c-1)v) \partial_v f_t(u, v) \\ & + \frac{i\hbar}{2} (\delta(c+1) \partial_u^2 f_t(u, v) + (\delta\delta' + c^2 - 1) \partial_u \partial_v f_t(u, v) + \delta'(c-1) \partial_v^2 f_t(u, v)). \end{aligned}$$

By setting $\Delta = e^t + e^{-t} - c(e^t - e^{-t})$, the solution is given by

$$(1.7) \quad :e_*^{\frac{t}{i\hbar}2u^{ov}}:_K = \frac{2}{\sqrt{\Delta^2 - (e^t - e^{-t})^2 \delta \delta'}} e^{\frac{1}{i\hbar} \frac{e^t - e^{-t}}{\Delta^2 - (e^t - e^{-t})^2 \delta \delta'}} ((e^t - e^{-t})(\delta' u^2 + \delta v^2) + 2\Delta uv).$$

(See [11] to know how to find this formula.)

Setting $\delta \delta' = \rho^2$, we see

$$(1.8) \quad \sqrt{\Delta^2 - (e^t - e^{-t})^2 \delta \delta'} = e^{-t} \sqrt{((1-c+\rho)e^{2t} + (1+c-\rho))((1-c-\rho)e^{2t} + (1+c+\rho))}.$$

As $:e_*^{z \frac{2}{i\hbar} u^{ov}}:_K$ has double branched singularities, we have to prepare two \pm sheets with slits. Hence, we have two origins 0_+ , 0_- for $:e_*^{z \frac{2}{i\hbar} u^{ov}}:_K$. Noting that $:e_*^{0 \frac{1}{i\hbar} 2u^{ov}}:_K = \sqrt{1}$, we set

$$(1.9) \quad :e_*^{0_+ \frac{1}{i\hbar} u^{ov}}:_K = 1, \quad :e_*^{0_- \frac{1}{i\hbar} u^{ov}}:_K = -1.$$

Note also that $:e_*^{\pm \pi i \frac{1}{i\hbar} 2u^{ov}}:_K = \sqrt{1}$, but this is *not* an absolute scalar. The \pm sign depends on K and the path from 0 to πi by setting $:e_*^{0 \frac{1}{i\hbar} 2u^{ov}}:_K = 1$. As this is a scalar-like element belonging to a one parameter subgroup, we call it a *q-scalar*.

If $t = \pm \frac{\pi i}{2}$, then (1.7) is called the **polar element** and denoted by ε_{00} [10], [11]:

$$(1.10) \quad :\varepsilon_{00}:_K = \frac{1}{\sqrt{c^2 - \delta \delta'}} e^{\frac{1}{i\hbar} \frac{-1}{c^2 - \delta \delta'} (\delta' u^2 + \delta v^2 - 2cuv)}.$$

For the simplest case $c = \delta = \delta' = 0$ in (1.7) is the **Weyl ordered expression**. This is not a generic ordered expression having singular points on the imaginary axis, and this is πi -alternating periodic.

On the contrary, the **unit ordered expression** is given by $K=I$, i.e. $\delta=\delta'=1$, $c=0$. By (1.7), we have

$$(1.11) \quad :e_*^{\frac{t}{i\hbar} 2u^{ov}}:_I = \frac{2}{\sqrt{4}} e^{\frac{1}{4i\hbar} (e^{2t} + e^{-2t} + 2)(u^2 + v^2) + 2(e^{2t} - e^{-2t})uv}.$$

This is πi -periodic and there is no singular point.

The case where $c=0$ and $\delta \delta' \neq 1$,

For the case $\delta = \delta' = 0$ but $c \neq 0$ which involves the **normal ordered expression** for $c = 1$, we see that

$$(1.12) \quad :e_*^{\frac{t}{i\hbar} 2u^{ov}}:_K = \frac{2}{\sqrt{\Delta^2}} e^{\frac{1}{i\hbar} \frac{e^t - e^{-t}}{\Delta^2} (2\Delta uv)} = \frac{2}{\Delta} e^{\frac{1}{i\hbar} \frac{e^t - e^{-t}}{\Delta} 2uv}.$$

This is the case where the singular points are not branching ones and they are sitting πi periodically on a single line parallel to the imaginary axis whose real part are given by $\log \left| \frac{c+1}{c-1} \right|$. We see also that $:e_*^{\frac{t}{i\hbar} 2u^{ov}}:_K$ is alternating πi -periodic along the imaginary axis.

There is another special expression parameter K such that $(1+c)^2 - \rho^2 = 0$, which will be called the **separating ordered expression**. In this case, we have

$$(1.13) \quad \sqrt{\Delta^2 - (e^t - e^{-t})^2 \delta \delta'} = \sqrt{4(1+c^2) - 4e^{2t}}.$$

This is the case where the periodicity of $:e_*^{(s+it) \frac{1}{i\hbar} 2u^{ov}}:_K$ w.r.t. t changes at $s \leq \log |4(1+c^2)|$.

1.1 Generic properties of *-exponential functions of quadratic forms

In the previous note [11] we have studied *-exponential functions of quadratic forms under a fixed expression parameter. In this chapter, we fix a quadratic form $\alpha+2u\circ v$ for $\alpha \in \mathbb{C}$, but move the expression parameter. The important view point here is that we are thinking that K is moving by the individual time parameter.

One parameter subgroup $e_*^{z(\alpha+2u\circ v)}$ of the *-exponential function of a quadratic form $\alpha+2u\circ v$ has remarkable properties that the periodicity depends on α and the expression parameters. We first summarize generic properties of the *-exponential function $:e_*^{z(\alpha+\frac{1}{i\hbar}2u\circ v)}:_K$ mentioned in [10], [11], where by generic property we mean properties held in almost all (open dense) expressions.

Note first that $\frac{1}{i\hbar}u\circ v - \frac{1}{2} = \frac{1}{i\hbar}u*v$, and the fundamental exponential law (cf. (d) below)

$$e_*^{z(\alpha+\frac{1}{i\hbar}2u\circ v)} = e^{z\alpha} e_*^{z\frac{1}{i\hbar}2u\circ v}, \quad e_*^{z(\alpha+\frac{1}{i\hbar}2u\circ v)} * e_*^{z'(\alpha+\frac{1}{i\hbar}2u\circ v)} = e_*^{(z+z')(\alpha+\frac{1}{i\hbar}2u\circ v)}.$$

Hence the essential part is $e_*^{z\frac{1}{i\hbar}2u\circ v}$, but its periodicity depends on α . If $\alpha=1$, then $e_*^{z\frac{1}{i\hbar}2u\circ v} = e_*^{z(-1+\frac{1}{i\hbar}2u\circ v)}$.

Let $Hol(\mathbb{C}^2)$ be the space of all holomorphic functions of $(u, v) \in \mathbb{C}^2$ with the uniform convergent topology on each compact subset.

(a) In generic ordered expressions, one may assume there is no singular point on the real axis and the pure imaginary axis.

(b) $e_*^{z\frac{2}{i\hbar}u\circ v}$ is a $Hol(\mathbb{C}^2)$ -valued $2\pi i$ -periodic function, i.e. $:e_*^{(z+2\pi i)\frac{2}{i\hbar}u\circ v}:_K = :e_*^{z\frac{2}{i\hbar}u\circ v}:_K$. More precisely, it is πi -periodic or alternating πi -periodic.

(c) $e_*^{z\frac{2}{i\hbar}u\circ v}$ is rapidly decreasing along any line parallel to the real line.

(d) As a result, $e_*^{z\frac{2}{i\hbar}u\circ v}$ must have periodic singular points. But the singular points are double branched. Hence $e_*^{z\frac{2}{i\hbar}u\circ v}$ is double-valued with the sign ambiguity. Singular point set Σ_K is distributed πi -periodically along the two lines parallel to the imaginary axis. In spite of double-valued nature, the exponential law

$$e_*^{z\frac{2}{i\hbar}u\circ v} * e_*^{z'\frac{2}{i\hbar}u\circ v} = e_*^{(z+z')\frac{2}{i\hbar}u\circ v}$$

holds under the calculation such that $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

(e) By requesting 1 at $z = 0$, i.e. $:e_*^{0\frac{2}{i\hbar}u\circ v}:_K = 1$, the value $:e_*^{[0\sim z]\frac{2}{i\hbar}u\circ v}:_K$ is determined uniquely, where $[0\sim z]$ is a path from 0 to z avoiding Σ_K and evaluating at z .

1.2 Periodicity and the exchanging interval

In generic ordered expressions $:e_*^{\pm\frac{z}{i\hbar}2u\circ v}:_K$, is $2\pi i$ -periodic along any line parallel to the imaginary axis. In generic K , there are two real number \hat{a}, \hat{b} ($\hat{a} < \hat{b}$) such that the set Σ_K of singular points are lying on the lines $(\hat{a}+i\mathbb{R}) \cup (\hat{b}+i\mathbb{R})$.

Assuming $(1-c)^2 - \rho^2 \neq 0$ in generic K , we set

$$a = \log \left| \frac{1+c-\rho}{1-c+\rho} \right|, \quad b = \log \left| \frac{1+c+\rho}{1-c-\rho} \right|$$

in (1.8). Then $\hat{a}=a\wedge b$, $\hat{b}=a\vee b$. The open interval $I_o(K) = (a\wedge b, a\vee b)$ is called the (sheet) **exchanging interval** of $e_*^{z\frac{1}{i\hbar}2u\circ v}$. As the exponential law shows that $e_*^{z\frac{1}{i\hbar}2u\circ v}$, $e_*^{z\frac{1}{i\hbar}2v*u}$ have singular points

on the same two lines, $I_o(K)$ is called also the exchanging interval of these. The separating ordered expression is the case where $a \wedge b$ is $-\infty$, i.e. $I_o(K) = (-\infty, a \vee b)$.

As the pattern of periodicity depends on how the circle $\{re^{i\theta}; \theta \in \mathbb{R}\}$ round the singular points, it depends delicately on K . There are three disjoint open subsets \mathfrak{K}_\pm and \mathfrak{K}_0 of the space of expression parameters such that $\mathfrak{K}_+ \cup \mathfrak{K}_- \cup \mathfrak{K}_0$ is dense.

(1) If $K \in \mathfrak{K}_+$ (resp. \mathfrak{K}_-), the singular set of $:e_*^{\frac{z}{i\hbar}2u^{ov}}:_K$ appears πi -periodically only in the open right (resp. left) half plane, along two lines parallel to the imaginary axis, and the $*$ -exponential functions form a complex semi-group over the left (resp. right) half plane without sign ambiguity by requesting 1 at $t=0$. Moreover, $:e_*^{\pm z \frac{1}{i\hbar}2u^{ov}}:_K$, is alternating πi -periodic on the imaginary axis, and $:e_*^{\pm \frac{z}{i\hbar}2u^{ov}}:_K$ is rapidly decreasing of $e^{-|z|}$ order along any line parallel to the real line.

In particular, we have $:e_*^{[0 \rightarrow \pi] \frac{i}{i\hbar}2u^{ov}}:_K = :e_*^{[0 \rightarrow \pi] \frac{-i}{i\hbar}2u^{ov}}:_K = -1$, and hence $\varepsilon_{00}^2 = -1$ by the $*$ -product, where $[0 \rightarrow a]$ is the path starting from the origin 0 ending at a along the line segment, but the $*$ -exponential is evaluated at $t=a$ by the continuous chase from 0 to a along the path $[0 \rightarrow a]$.

The special ordered expression K_s used in [10] is in \mathfrak{K}_+ .

(2) If $K \in \mathfrak{K}_0$, singular set appears in both left and right half-planes, but not on the imaginary axis. Both of these lines are parallel to the imaginary axis. In particular, we have $:e_*^{[0 \rightarrow \pi] \frac{i}{i\hbar}2u^{ov}}:_K = 1$, and then $\varepsilon_{00}^2 = 1$ by the natural product along the imaginary axis. Moreover, $:e_*^{\pm \frac{z}{i\hbar}2u^{ov}}:_K$, is πi -periodic on the imaginary axis.

The Siegel ordered expression mentioned in [10], [11], i.e. $\text{Re} \frac{1}{\hbar} \langle \xi(iK), \xi \rangle \geq c_K |\xi|^2$ for some $c_K > 0$, is in \mathfrak{K}_0 .

For every $s \in I_o(K)$, by replacing $\frac{2}{i\hbar}u^{ov}$ to $\frac{1}{i\hbar}u^{ov}$, $:e_*^{(s+it)\frac{1}{i\hbar}u^{ov}}:_K$ is 2π -periodic w.r.t. t , but $:e_*^{(s+it)\frac{1}{i\hbar}u^{*v}}:_K$ is alternating 2π -periodic w.r.t. t . On the contrary, for every $s \in \mathbb{R} \setminus \overline{I_o(K)}$, $:e_*^{(s+it)\frac{1}{i\hbar}u^{ov}}:_K$ is alternating 2π -periodic w.r.t. t , and $:e_*^{(s+it)\frac{1}{i\hbar}u^{*v}}:_K$ is 2π -periodic w.r.t. t .

1.2.1 Rules of setting slits and evaluations

As there is no singular point on the real axis, and the pure imaginary axis in generic ordered expression, one can evaluate by (1.9)

$$:e_*^{(s_+)(\frac{1}{i\hbar}u^{ov}+\alpha)}:_K, \quad :e_*^{(s_-)(\frac{1}{i\hbar}u^{ov}+\alpha)}:_K, \quad :e_*^{(is_+)(\frac{1}{i\hbar}u^{ov}+\alpha)}:_K, \quad :e_*^{(is_-)(\frac{1}{i\hbar}u^{ov}+\alpha)}:_K$$

univalent way for every $s \in \mathbb{R}$, We see

$$(1.14) \quad :e_*^{(s_-)(\frac{1}{i\hbar}u^{ov}+\alpha)}:_K = -:e_*^{(s_+)(\frac{1}{i\hbar}u^{ov}+\alpha)}:_K, \quad :e_*^{(is_-)(\frac{1}{i\hbar}u^{ov}+\alpha)}:_K = -:e_*^{(is_+)(\frac{1}{i\hbar}u^{ov}+\alpha)}:_K$$

where (s_+) , (s_-) are s in the positive, the negative sheet respectively.

We want to evaluate $:e_*^{(s_\pm+it)(\frac{1}{i\hbar}u^{ov}+\alpha)}:_K$ by solving the differential equation

$$\frac{d}{dt}f_t = :i(\frac{1}{i\hbar}u^{ov}+\alpha):_K *_K f_t, \quad f_0 = :e_*^{(s_\pm)(\frac{1}{i\hbar}u^{ov}+\alpha)}:_K.$$

If we neglect the sign ambiguity, the solution is given by $:e_*^{(s_\pm+it)(\frac{1}{i\hbar}u^{ov}+\alpha)}:_K$. As singular points are double branched singular points, we have to mind the slits where sheets are changing.

There are many ways to set slits. A typical way is to set slits 2π -periodically between singular points, if $\pm\infty$ split in the Riemann surface. In this case, we have two $-\infty_{\pm}$, and two ∞_{\pm} , where $\pm\infty_{+}$ are in positive sheet and $\pm\infty_{-}$ are in negative sheet. This will be used for the case $\alpha \in \mathbb{Z} + \frac{1}{2}$.

Another is to set slits between $\pm\infty$ and singular points, where $\pm\infty$ do not split in the Riemann surface. This will be used for the case $\alpha \in \mathbb{Z}$. In any case, the value $:e_*^{z\frac{1}{i\hbar}u\circ v+\alpha}:_K$ changes sign discontinuously when a point z crossed a slit in the same sheet. One has to change sheets for the continuous tracing.

As a standard one, we set slits between singular points just as in the vertical segments in the picture below. We have two zero 0_{+} , 0_{-} and two $-\infty$; $-\infty_{+}$, $-\infty_{-}$ and two ∞ ; ∞_{+} , ∞_{-} . The $*$ -exponential function $:e_*^{z(\alpha+\frac{1}{i\hbar}2u\circ v)}:_K$ is viewed as a single $Hol(\mathbb{C}^2)$ -valued function on this space.

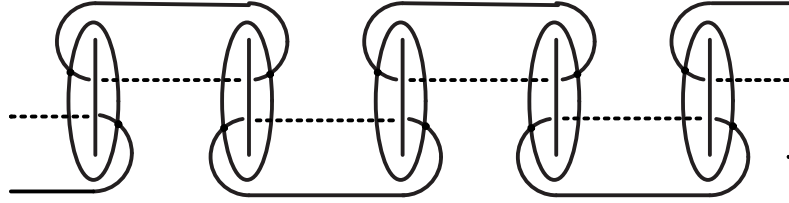


Fig.1 Riemann surface of infinite genus

where vertical lines in Fig.1 are near to parallel to the real axis.

A little care is requested to apply the Cauchy's integral theorem. All closed paths drawn in Fig.1 give closed paths in the Riemann surface of infinite genus, where dotted lines are in the negative sheets. On such a curve, we have to evaluate by continuous tracing along a curve in spite of the evaluation rule (1.14). Note that (1.14) is the evaluation along a path from 0_{\pm} without crossing slits.

2 Elements defined by integrals

Note that in a generic ordered expression $e_*^{\pm\frac{t}{i\hbar}2u\circ v}$ is rapidly decreasing with the growth order $e^{-|t|}$ along lines parallel to the real axis. Noting $v*u=u\circ v+\frac{1}{2}i\hbar$, we see the following:

Proposition 2.1 *In generic ordered expressions such that there is no singular point on the real axis*

$$\lim_{t \rightarrow \infty} e_*^{t\frac{1}{i\hbar}2u*v} = 0, \quad \lim_{t \rightarrow -\infty} e_*^{t\frac{1}{i\hbar}2v*u} = 0,$$

but the following limit exists

$$\lim_{t \rightarrow -\infty} e_*^{t\frac{1}{i\hbar}2u*v} = \varpi_{00}, \quad \lim_{t \rightarrow \infty} e_*^{t\frac{1}{i\hbar}2v*u} = \overline{\varpi}_{00}.$$

More precisely, in a fixed generic expression parameter $K = \begin{bmatrix} \delta & c \\ c & \delta' \end{bmatrix}$, $:e_*^{t\frac{1}{i\hbar}2u\circ v}:_K$ is smooth rapidly decreasing in \pm directions and (1.7) gives

$$\begin{aligned} :\varpi_{00}:_K &= \lim_{t \rightarrow -\infty} :e_*^{t\frac{1}{i\hbar}2u*v}:_K = \frac{2}{\sqrt{(1+c)^2 - \delta\delta'}} e^{-\frac{1}{i\hbar} \frac{1}{(1+c)^2 - \delta\delta'} (\delta u^2 - (1+c)2uv + \delta' v^2)}, \\ (2.1) \quad :\overline{\varpi}_{00}:_K &= \lim_{t \rightarrow \infty} :e_*^{t\frac{1}{i\hbar}2v*u}:_K = \frac{2}{\sqrt{(1-c)^2 - \delta\delta'}} e^{\frac{1}{i\hbar} \frac{1}{(1-c)^2 - \delta\delta'} (\delta u^2 + (1-c)2uv + \delta' v^2)}, \\ \lim_{t \rightarrow \infty} :e_*^{t\frac{1}{i\hbar}2u*v}:_K &= 0, \quad \lim_{t \rightarrow -\infty} :e_*^{t\frac{1}{i\hbar}2v*u}:_K = 0. \end{aligned}$$

without sign ambiguity. We call ϖ_{00} and $\overline{\varpi}_{00}$ **vacuum** and **bar-vacuum** respectively. These are contained in the space $\mathbb{C}e^{Q(u,v)}$ of exponential functions of quadratic forms.

As ϖ_{00} is defined by the limit, we see $u*v*\varpi_{00} = 0 = \varpi_{00}*u*v$. But the “bumping identity” $v*f(u*v)=f(v*u)*v$ give the following:

Lemma 2.1 $v*\varpi_{00}=0=\varpi_{00}*u$, $u*\overline{\varpi}_{00}=0=\overline{\varpi}_{00}*v$ in generic ordered expressions.

Proof Using the continuity of $v*$, we see that $v*\lim_{t \rightarrow -\infty} e_*^{t\frac{1}{i\hbar}2u*v} = \lim_{t \rightarrow -\infty} v*e_*^{t\frac{1}{i\hbar}2u*v}$. Hence, the bumping identity gives $\lim_{t \rightarrow -\infty} e_*^{t\frac{1}{i\hbar}2v*u}*v=0$ by using Proposition 2.1. \square .

The exponential law gives

$$\varpi_{00}*\varpi_{00}=\varpi_{00}, \quad \overline{\varpi}_{00}*\overline{\varpi}_{00}=\overline{\varpi}_{00}.$$

However, the product $\varpi_{00}*\overline{\varpi}_{00}$ diverges whenever this is defined as $\lim_{t \rightarrow -\infty} e_*^{tu*v}*\overline{\varpi}_{00}$. But in fact, it depends how the product is defined, as it will be mentioned below it is better so to define that $\varpi_{00}*\overline{\varpi}_{00} = 0$.

The next identities are easy to see

$$\varpi_{00}*\frac{1}{i\hbar}u\circ v = \frac{1}{2}\varpi_{00}, \quad \frac{1}{i\hbar}u\circ v*\overline{\varpi}_{00} = -\frac{1}{2}\overline{\varpi}_{00}.$$

Note that in order to keep the associativity

$$(2.2) \quad (\varpi_{00}*\frac{1}{i\hbar}u\circ v)*\overline{\varpi}_{00} = \varpi_{00}*(\frac{1}{i\hbar}u\circ v*\overline{\varpi}_{00}),$$

we have to define

$$\frac{1}{2}\varpi_{00}*\overline{\varpi}_{00} = -\frac{1}{2}\varpi_{00}*\overline{\varpi}_{00} = 0.$$

Note that there is no sign ambiguity. But strictly speaking, vacuums should be so defined carefully that they do not involve sign ambiguity.

Note In the traditional ring theory, vacuums are defined as maximal left ideals. But we prefer to use the notion of vacuums as physicists uses, to which we can not give a mathematical definition, because we do not know what is the true nature of the *vacuum*. Equality in Lemma 2.1 may be understood to give the separation of a “configuration space” from the phase space. It should be noted also that we have two sheets and these changes sign in the opposite sheet.

2.1 Vacuums and pseudo-vacuums

Note that if $|\operatorname{Re} t|$ is sufficiently large, then $e_*^{(t+i\sigma)\frac{1}{i\hbar}u*v}$, $e_*^{(t+i\sigma)\frac{1}{i\hbar}v*u}$ are both 2π -periodic w.r.t. σ . Thus, it is better to define vacuums as the limits of period integral:

$$(2.3) \quad 2\pi\varpi_{00} = \lim_{t \rightarrow -\infty} \int_{-\pi}^{\pi} e_*^{(t+i\sigma)\frac{1}{i\hbar}u*v} d\sigma, \quad 2\pi\overline{\varpi}_{00} = \lim_{s \rightarrow \infty} \int_{-\pi}^{\pi} e_*^{(s+i\sigma)\frac{1}{i\hbar}v*u} d\sigma.$$

In fact, we have no need to take the limit since these are constant if $|t|, |s|$ are sufficiently large. In fact, Cauchy's integral theorem together with 2π -periodicity gives

$$\begin{aligned}\frac{1}{2\pi} \int_{-\pi}^{\pi} :e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v}:_K d\sigma &= \begin{cases} :\varpi_{00}:_K, & s < a \\ 0, & s > b \end{cases} \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} :e_*^{(s+i\sigma)\frac{1}{i\hbar}v*u}:_K d\sigma &= \begin{cases} 0, & s < a \\ :\overline{\varpi}_{00}:_K, & s > b. \end{cases}\end{aligned}$$

The product $\varpi_{00}*\overline{\varpi}_{00}$ can not be defined directly by the definition, but the product

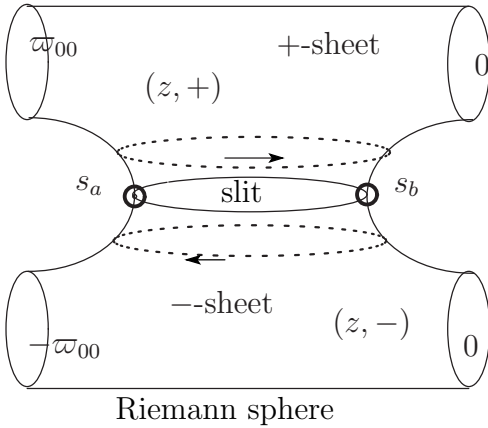
$$\int_{-\pi}^{\pi} e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v} d\sigma * \int_{-\pi}^{\pi} e_*^{(s'+i\sigma')\frac{1}{i\hbar}v*u} d\sigma' = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v+(s'+i\sigma')\frac{1}{i\hbar}v*u} d\sigma d\sigma'$$

can be defined always to give 0, for by using $\frac{1}{i\hbar}u*v = \frac{1}{i\hbar}u\circ v - \frac{1}{2}$, and $\frac{1}{i\hbar}v*u = \frac{1}{i\hbar}u\circ v + \frac{1}{2}$, the change of variables gives

$$\int_{-\pi}^{\pi} e^{+i\sigma} d\sigma \int_{-\pi}^{\pi} e^{\frac{1}{2}(s'-s-i\tau)} e_*^{(s+s'+i\tau)\frac{1}{i\hbar}u\circ v} d\tau = 0.$$

Thus, we have

Proposition 2.2 *For every polynomial $p(u, v)$, $\varpi_{00}*p_*(u, v)*\overline{\varpi}_{00}=0=\overline{\varpi}_{00}*p(u, v)*\varpi_{00}$ in generic ordered expression. (Cf. (2.2).)*



Keeping the periodicity in mind, the Riemann surface of $e_*^{z\frac{1}{i\hbar}u*v}$ may be viewed as the Riemann sphere, where s_a, s_b in the figure are singular points. Vertical circuit between these correspond to the lines parallel to the imaginary axis. The complex number $z \in \mathbb{C}$ is expressed in the figure by $(z, +)$ and $(z, -)$ in \pm -sheet respectively. Note also that the integral along a such vertical circuit vanishes by the alternating periodicity.

In such a compact Riemann surface, homological cycles play important role. But some of them are not related to closed one parameter subgroups of $e_*^{t(\alpha+\frac{1}{i\hbar}u\circ v)}$.

Noting $:e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v}:_K$ is 4π -periodic for $|s| \gg 0$, and $\int_{-2\pi}^{2\pi} :e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v}:_K d\sigma = 0$ for $s \in I_o(K)$ by the alternating 2π -periodicity, we see

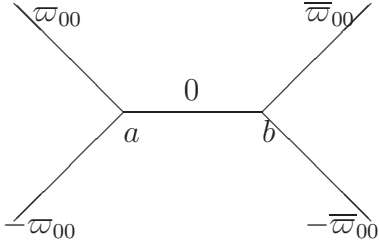
$$(2.4) \quad \begin{aligned}\frac{1}{4\pi} \int_{-2\pi}^{2\pi} :e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v}:_K d\sigma &= \begin{cases} :\varpi_{00}:_K, & s < a \\ 0, & a < s \end{cases} \\ \frac{1}{4\pi} \int_{-2\pi}^{2\pi} :e_*^{(s+i\sigma)\frac{1}{i\hbar}v*u}:_K d\sigma &= \begin{cases} 0, & s < b \\ :\overline{\varpi}_{00}:_K, & b < s, \end{cases}\end{aligned}$$

Hence we see

$$\frac{1}{i\hbar}u*v*\frac{1}{4\pi} \int_{-2\pi}^{2\pi} :e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v}:_K d\sigma = -\delta(s-a)\varpi_{00}.$$

It is easy to see that first equality of (2.4) is the solution of

$$\frac{d}{ds}f(s) = -\delta(s-a)\varpi_{00}*f(s), \quad f(-\infty) = \varpi_{00}.$$



Minding the negative sheet, the integral

$$\frac{1}{4\pi} \int_{-2\pi}^{2\pi} : \left(e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v} + e_*^{(s+i\sigma)\frac{1}{i\hbar}v*u} \right) :_K d\sigma$$

may be expressed symbolically by the l.h.s. picture. Note that if $a < s < b$, then $:e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v}:_K$, $\sigma \in [-\pi, \pi]$ is not a closed path.

If $a < s < b$, the integral on half-period depends on s

$$(2.5) \quad \frac{d}{ds} \int_0^{2\pi} e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v} d\sigma = \int_0^{2\pi} (-i) \frac{d}{d\sigma} e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v} d\sigma = \frac{1}{i} (e_*^{(s+i2\pi)\frac{1}{i\hbar}u*v} - e_*^{s\frac{1}{i\hbar}u*v}) = 2ie_*^{s\frac{1}{i\hbar}u*v}.$$

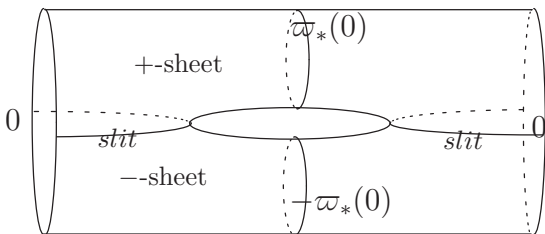
2.1.1 The pseudo-vacuum

Let $I_0(K) = [a, b]$ be the exchanging interval of $:e_*^{z\frac{1}{i\hbar}u\circ v}:_K$. The periodicity of $e_*^{(s+it)\frac{1}{i\hbar}u\circ v}$ with respect to t depends on $s \in \mathbb{R}$. $e_*^{(s+it)\frac{1}{i\hbar}u\circ v}$ is $2\pi i$ -periodic in t if $a < s < b$, and alternating $2\pi i$ -periodic, if $s < a$ or $b < s$. For $a < s < b$, we set

$$(2.6) \quad :\varpi_*(s):_K = \frac{1}{2\pi} \int_0^{2\pi} :e_*^{(s+it)(\frac{1}{i\hbar}u\circ v)}:_K dt.$$

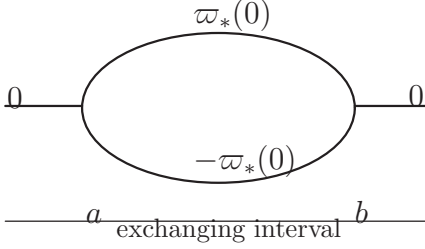
This is independent of s whenever $a < s < b$ and $(\frac{1}{i\hbar}u\circ v)* \int_0^{2\pi} e_*^{(s+it)\frac{1}{i\hbar}u\circ v} = 0$.

Suppose $K \in \mathfrak{K}_0$. Then $e_*^{it\frac{1}{i\hbar}u\circ v}$ is 2π -periodic and $\frac{1}{2\pi} \int_0^{2\pi} :e_*^{it(\frac{1}{i\hbar}u\circ v)}:_K dt$ has the idempotent property. It is easy to see that $\frac{1}{2\pi} \int_0^{2\pi} :e_*^{it(\frac{1}{i\hbar}u\circ v)}:_K dt = \frac{1}{4\pi} \int_0^{4\pi} :e_*^{it(\frac{1}{i\hbar}u\circ v)}:_K dt$.



This is called the **pseudo-vacuum**. We denote this by $:\varpi_*(0):_K$, but note that the pseudo-vacuum is expressed only by expression parameter $K \in \mathfrak{K}_0$. Minding the existence of opposite sheet and the $2\pi i$ -alternating periodicity for $|s| \gg 0$, we set the slits between singular points and $\pm\infty$. Consider now the integral $\frac{1}{4\pi} \int_0^{4\pi} :e_*^{(s+it)(\frac{1}{i\hbar}u\circ v)}:_K dt$. Putting the periodicity in mind, Cauchy's integral theorem gives

$$(2.7) \quad \frac{1}{4\pi} \int_0^{4\pi} :e_*^{(s+it)(\frac{1}{i\hbar}u\circ v)}:_K dt = \begin{cases} 0, & s < a \\ :\varpi_*(0):_K, & a < s < b, \\ 0, & b < s \end{cases} \quad K \in \mathfrak{K}_0$$



This formula may be expressed symbolically by the l.h.s. picture. Since $\varpi_{00}, \overline{\varpi}_{00}$ are defined respectively by integrals $\frac{1}{2\pi} \int_{-\pi}^{\pi} :e_*^{(s+i\sigma)(\frac{1}{i\hbar}u\circ v \mp \frac{1}{2})}:_K d\sigma$, for $s \ll 0$ and $s \gg 0$, the direct computation via changing variables gives also

$$\begin{aligned} :\varpi_{00}*\varpi_*(0):_K &= 0 = :\varpi_*(0)*\varpi_{00}:_K, & K \in \mathfrak{K}_0, \\ :\overline{\varpi}_{00}*\varpi_*(0):_K &= 0 = :\varpi_*(0)*\overline{\varpi}_{00}:_K, & K \in \mathfrak{K}_0. \end{aligned}$$

Note that $\varpi_{00}, \overline{\varpi}_{00}$ are defined also by $\frac{1}{4\pi} \int_{-2\pi}^{2\pi} :e_*^{(s+i\sigma)(\frac{1}{i\hbar}u\circ v \mp \frac{1}{2})}:_K d\sigma$, for $s \ll 0$ and $s \gg 0$. Using these we have also

Proposition 2.3 *If $K \in \mathfrak{K}_0$, then for every polynomial $p(u, v)$,*

$$\varpi_*(0)*p_*(u, v)*\varpi_{00}=0=\varpi_{00}*p_*(u, v)*\varpi_*(0), \quad \varpi_*(0)*p_*(u, v)*\overline{\varpi}_{00}=0=\overline{\varpi}_{00}*p_*(u, v)*\varpi_*(0).$$

Proof Note first that $(u\circ v)*\varpi_*(0)=0=\varpi_*(0)*(u\circ v)$, and $\varpi_{00}*u=0=v*\varpi_{00}$. Hence we have only to show

$$\varpi_*(0)*u*\varpi_{00} = 0 = \varpi_{00}*v*\varpi_*(0).$$

The bumping identity gives $\varpi_*(0)*u = u*\frac{1}{4\pi} \int_{-2\pi}^{2\pi} e_*^{t(\frac{1}{i\hbar}u\circ v+1)} dt$. Thus, for $s \ll 0$,

$$16\pi^2 \varpi_*(0)*u*\varpi_{00} = u* \int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} e_*^{it(\frac{1}{i\hbar}u\circ v+1)} *_K e_*^{(s+i\tau)\frac{1}{i\hbar}u\circ v} dt d\tau$$

Exponential law and changing variables make this integral vanish. \square

2.1.2 No other idempotent element

In general, $:e_*^{it\frac{1}{i\hbar}u\circ v}:_K$ is 4π -periodic one parameter subgroup in $t \in \mathbb{R}$ under a generic ordered expression. Hence the exponential law gives for every rational number $\frac{q}{p}$, $:e_*^{it(\frac{1}{i\hbar}u\circ v \pm \frac{q}{p})}:_K$ is $4p\pi$ -periodic one parameter subgroup and the period integral

$$\varpi_*(\pm \frac{q}{p}) = \frac{1}{4p\pi} \int_0^{4p\pi} e_*^{it(\frac{1}{i\hbar}u\circ v \pm \frac{q}{p})} dt$$

has the idempotent property $\varpi_*(\pm \frac{q}{p})*\varpi_*(\pm \frac{q}{p}) = \varpi_*(\pm \frac{q}{p})$. However, we show in what follows that this is nontrivial only if $\frac{q}{p}=0$.

Cauchy's integral theorem shows that if $0 < a$, then $:\varpi_*(\frac{q}{p}):_K = 0$ for $\frac{q}{p} > -\frac{1}{2}$, and if $b < 0$, then $:\varpi_*(\frac{q}{p}):_K = 0$ for $\frac{q}{p} < \frac{1}{2}$. Recalling (2.19), we see that pseudo-vacuum appears only for the case $a < 0 < b$, and the case $|\frac{q}{p}| \leq \frac{1}{2}$ is essential by a suitable shift of integers via bumping identity.

If $a < 0 < b$, i.e. $K \in \mathfrak{K}_0$, then $:e_*^{i\sigma(\frac{1}{i\hbar}u\circ v + \frac{q}{p})}:_K$ is $2p\pi$ -periodic, but if p is an even integer (hence q is an odd integer), then $:e_*^{i\sigma(\frac{1}{i\hbar}u\circ v + \frac{q}{p})}:_K$ is alternating $p\pi$ periodic, and

$$(2.8) \quad \frac{1}{2p\pi} \int_0^{2p\pi} :e_*^{(s+i\sigma)(\frac{1}{i\hbar}u\circ v + \frac{q}{p})}:_K d\sigma = 0.$$

In fact, we see a stronger result as follows:

Theorem 2.1 If $-\frac{1}{2} < \frac{q}{p} < \frac{1}{2}$, then under the K -expression, $\varpi_*(\pm \frac{q}{p}) = \frac{1}{2p\pi} \int_0^{2p\pi} e_*^{it(\frac{1}{i\hbar}u^{ov} \pm \frac{q}{p})} dt = 0$ except the case $\frac{q}{p} = 0$, where $\varpi_*(0) = \frac{1}{4\pi} \int_0^{4\pi} e_*^{it(\frac{1}{i\hbar}u^{ov})} dt = \frac{1}{2\pi} \int_0^{2\pi} e_*^{it(\frac{1}{i\hbar}u^{ov})} dt$.

Proof $e_*^{it(\frac{q}{p} + \frac{1}{i\hbar}u^{ov})}$ is $2p\pi$ -periodic, As $e_*^{it(\frac{q}{p} + \frac{1}{i\hbar}u^{ov})}$ and $e^{it\frac{1}{p}\tau}$ are $2p\pi$ -periodic, the Fourier expansion of this is given by

$$e_*^{i\tau(\frac{q}{p} + \frac{1}{i\hbar}u^{ov})} = \frac{1}{2p\pi} \sum_{k \in \mathbb{Z}} \int_0^{2p\pi} e_*^{it(\frac{q}{p} + \frac{1}{i\hbar}u^{ov} - \frac{k}{p})} dt e^{i\frac{k}{p}\tau}.$$

This converges in the C^∞ -topology on S^1 . Set the r.h.s. by $f(\tau) = \sum a_k e^{i\frac{k}{p}\tau}$. Note the $2p\pi$ -periodicity, and also that $f(\tau)$ has the property $f(\tau + 2\pi) = e^{2\pi i \frac{q}{p}} f(\tau)$. This gives

$$f(\tau + 2\pi) = \sum a_k e^{i\frac{k}{p}2\pi} e^{i\frac{k}{p}\tau} = e^{2\pi i \frac{q}{p}} \sum a_k e^{i\frac{k}{p}\tau}.$$

The uniqueness of Fourier coefficients gives $a_k e^{i\frac{k}{p}2\pi} = e^{2\pi i \frac{q}{p}} a_k$ and hence a_k vanishes if $k \neq q + p\ell$, $\ell \in \mathbb{Z}$. Hence the Fourier series of $f(\tau)$ is

$$(2.9) \quad e_*^{i\tau(\frac{q}{p} + \frac{1}{i\hbar}u^{ov})} = \sum_{\ell} \frac{1}{2p\pi} \int_0^{2p\pi} e_*^{it(\frac{1}{i\hbar}u^{ov} - \ell)} dt e^{i(\frac{q}{p} + \ell)\tau} = \sum_{\ell} \frac{1}{2\pi} \int_0^{2\pi} e_*^{it(\frac{1}{i\hbar}u^{ov} - \ell)} dt e^{i(\frac{q}{p} + \ell)\tau},$$

for the integrand is 2π -periodic. Componentwise integration gives that if $\frac{q}{p} + \ell \neq 0$, then

$$\varpi_*\left(\frac{q}{p}\right) = \frac{1}{2p\pi} \int_0^{2p\pi} e_*^{i\tau(\frac{q}{p} + \frac{1}{i\hbar}u^{ov})} d\tau = \sum_{\ell} \frac{1}{2pi} \int_0^{2\pi} e_*^{it(\frac{1}{i\hbar}u^{ov} - \ell)} dt \int_0^{2p\pi} e^{i(\frac{q}{p} + \ell)\tau} d\tau = 0.$$

This is nontrivial if and only if $\frac{q}{p} = 0$ and $\ell = 0$. □

2.2 Two inverses in generic ordered expressions

Recall the formula (1.7) shows that $:e_*^{\zeta \frac{1}{i\hbar}u^{ov}}:_K$ is $4\pi i$ -periodic along any line parallel to the pure imaginary axis, and rapidly decreasing along any line parallel to the real line.

Similar to Jacobi's theta function in [9], we have a convergence of the infinite summation

$$(2.10) \quad \Theta_L(\zeta; *) = \sum_{k=-\infty}^{\infty} e_*^{(\zeta + kL) \frac{1}{i\hbar}u^{ov}}, \quad \Theta_L(\zeta; K) = \sum_{k=-\infty}^{\infty} :e_*^{(\zeta + kL) \frac{1}{i\hbar}u^{ov}}:_K$$

whenever there is no singular point on the line $\zeta + \mathbb{R}$. Hence we have two different inverses of $1 - e_*^{\zeta \frac{1}{i\hbar}u^{ov}}$:

$$\sum_{k=0}^{\infty} e_*^{(\zeta + kL) \frac{1}{i\hbar}u^{ov}}, \quad \sum_{k=-\infty}^{-1} e_*^{(\zeta + kL) \frac{1}{i\hbar}u^{ov}}.$$

At the first glance, $\Theta_L(\zeta; *)$ looks to be double periodic such that

$$\Theta_L(\zeta + L; *) = \Theta_L(\zeta; *), \quad \Theta_L(\zeta + 4\pi i; *) = \Theta_L(\zeta; *).$$

However, there is no rule to evaluate $\Theta_L(\zeta; *)$ at $\zeta = x+iy$ univalent way if $\zeta = x+iy+kL$ is on the slit. Hence, $\Theta_L(\zeta; *)$ is well-defined only in the case that the exchanging interval $I_o(K)$ is empty (eg. the case $\rho = 0$ in (1.12)). If this is the case, $\Theta_L(\zeta; *)$ is a double periodic element having essential singular points. Thus, we have to develop some other theory for such an “elliptic” element, which will be discussed in forthcoming paper.

Similarly, supposing there is no singular point on $iy+\mathbb{R}$, (1.7) gives for $|\operatorname{Re} z| < \frac{1}{2}$ that

$$D(iy, z + \frac{1}{i\hbar}u \circ v, K) = \int_{-\infty}^{\infty} :e_*^{(x+iy)(z + \frac{1}{i\hbar}u \circ v)}:_K dx$$

is an element of $Hol(\mathbb{C}^2)$ in generic K . By Cauchy’s integral theorem $D(iy, z + \frac{1}{i\hbar}u \circ v, K)$ does not depend on iy if $|y|$ is sufficiently small. But, the value jumps discontinuously when $iy+\mathbb{R}$ hits the singularity.

Note that integrals $\int_{-\infty}^0 e_*^{t(z + \frac{1}{i\hbar}u \circ v)} dt$, $-\int_0^{\infty} e_*^{t(z + \frac{1}{i\hbar}u \circ v)} dt$ give inverses of $z + \frac{1}{i\hbar}u \circ v$ in generic ordered expressions, which are denoted by $(z + \frac{1}{i\hbar}u \circ v)_{*+}^{-1}$, $(z + \frac{1}{i\hbar}u \circ v)_{*-}^{-1}$ respectively. The following may be viewed as a Sato’s hyperfunction:

Proposition 2.4 *If $-\frac{1}{2} < \operatorname{Re} z < \frac{1}{2}$, then the difference of the two inverses is given by*

$$(2.11) \quad (z + \frac{1}{i\hbar}u \circ v)_{*+}^{-1} - (z + \frac{1}{i\hbar}u \circ v)_{*-}^{-1} = \int_{-\infty}^{\infty} e_*^{t(z + \frac{1}{i\hbar}u \circ v)} dt,$$

which is holomorphic on this strip in generic ordered expressions. This will be called $$ -delta function and denoted by $\delta_*(-iz + \frac{1}{\hbar}u \circ v)$.*

Note also that

$$:\delta_*(-iz + \frac{1}{\hbar}u \circ v):_K = D(0, z + \frac{1}{i\hbar}u \circ v, K).$$

Recall that $(z + \frac{1}{i\hbar}u \circ v) * \delta_*(-iz + \frac{1}{\hbar}u \circ v) = 0$. Thus, we have to set

$$e_*^{is(z + \frac{1}{i\hbar}u \circ v)} * \delta_*(-iz + \frac{1}{\hbar}u \circ v) = \delta_*(-iz + \frac{1}{\hbar}u \circ v)$$

whenever the $*$ -product $e_*^{isH} * f$ is defined as the real analytic solution of the evolution equation $\frac{d}{ds}f_s = iH * f_s$, $f_0 = f$. In spite of this, the integral element $D(is, z + \frac{1}{i\hbar}u \circ v, K)$ is not continuous in s in general.

For a, b such that $a \neq b$, four elements

$$(2.12) \quad \frac{1}{b-a} \left\{ (a + \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1} - (b + \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1} \right\} \quad (\text{independent } \pm),$$

give respectively inverses of

$$(a + \frac{1}{i\hbar}u \circ v) * (b + \frac{1}{i\hbar}u \circ v).$$

Thus, we define the $*$ -product $(a + \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1} * (b + \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1}$ by this formula. Note that $*$ -multiplications are calculated by summations. Similarly, we define

$$(a + \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1} * (a + \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1} = -\frac{d}{da} (a + \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1}, \quad (\pm \text{ respectively})$$

but $(a + \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1} * (a + \frac{1}{i\hbar}u \circ v)_{*\mp}^{-1}$ diverges. We refer such calculations to *resolvent calculus*. (Cf.[9].)

On the other hand, note that a change of variables $t \rightarrow -t$ gives

$$((-z) + \frac{1}{i\hbar}u \circ v)_{*-}^{-1} = - \int_0^\infty e_*^{-t(z - \frac{1}{i\hbar}u \circ v)} dt = - \int_{-\infty}^0 e_*^{(z - \frac{1}{i\hbar}u \circ v)} dt.$$

Thus, in generic ordered expressions, we see that

$$(2.13) \quad (z - \frac{1}{i\hbar}u \circ v)_{*-}^{-1} = -((-z) + \frac{1}{i\hbar}u \circ v)_{*-}^{-1}.$$

Proposition 2.5 *In generic ordered expressions, integrals $\int_{-\infty}^0 e_*^{t(z + \frac{1}{i\hbar}u \circ v)}$ and $\int_{-\infty}^0 e_*^{t(z - \frac{1}{i\hbar}u \circ v)}$ converge on the domain $\operatorname{Re} z > -\frac{1}{2}$.*

Using this, note first that $(z \pm \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1}$ are holomorphic on the domain $\operatorname{Re} z > -\frac{1}{2}$ in generic ordered expressions. It is natural to expect that

$$(z \pm \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1} = C(C(z \pm \frac{1}{i\hbar}u \circ v))_{*\pm}^{-1}$$

for any non-zero constant C . But this holds only for $C = re^{i\theta}$ such that $|\theta|$ is small enough, because of the singular point of $e_*^{t(z + \frac{1}{i\hbar}u \circ v)}$. To confirm this, we set $C = re^{i\theta}$ and consider the integral

$$re^{i\theta} \int_{-\infty}^0 e_*^{re^{i\theta}t(z \pm \frac{1}{i\hbar}u \circ v)} dt.$$

It is easy to see that its r -derivative vanishes for $r > 0$. For the θ -derivative, note that in generic K -ordered expressions, the phase part of the integrand is bounded in t and the amplitude is

$$\frac{2e^{i\theta}tz}{(1-\kappa)e^{e^{i\theta}t/2} + (1+\kappa)e^{-e^{i\theta}t/2}}, \quad \kappa \neq 1.$$

The integral converges whenever $\operatorname{Re} e^{i\theta}(z \pm \frac{1}{2}) > 0$, and $-te^{i\theta}$ does not hit the singular points. Then, the integration by parts gives that its θ -derivative vanishes.

It follows that in generic ordered expressions, $(z \pm \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1}$ are holomorphic on an overhanged sectoral domain

$$\mathbb{C} \setminus \{-D - \frac{1}{2}\}, \quad D = \{z = re^{i\theta}, |\theta| < \frac{\pi}{2} - \varepsilon\}.$$

Remark If $a > 0$, then $a((a(z \pm \frac{1}{i\hbar}u \circ v)_{*\pm})^{-1}) = (z \pm \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1}$, but $a((a(z \pm \frac{1}{i\hbar}u \circ v)_{*\pm})^{-1}) = (z \pm \frac{1}{i\hbar}u \circ v)_{*\mp}^{-1}$ for $a < 0$.

Next, it is natural to expect that the bumping identity $(u \circ v) * v = v * (u \circ v - i\hbar)$ gives the following “sliding identities”

$$v_{*+}^{-1} * (z + \frac{1}{i\hbar}u \circ v)_{*+}^{-1} * v = (z - 1 + \frac{1}{i\hbar}u \circ v)_{*+}^{-1}, \quad v_{*+}^{-1} * (z - \frac{1}{i\hbar}u \circ v)_{*-}^{-1} * v = (z + 1 - \frac{1}{i\hbar}u \circ v)_{*-}^{-1}$$

whenever one can use the inverse of v in a suitable ordered expression. In § 4.2.1, analytic continuation will be produced via these sliding identities.

However, the existence of v_{*+}^{-1} is not a generic property. In this note, we use the sliding identity by using, instead of v_{*+}^{-1} , the left inverse v° of v given by (2.14).

Remark There is a K -ordered expression such that $\int_{-\infty}^0 e_*^{tv} dt \cdot_K$ converges to give an inverse of $:v_{*+}^{-1} \cdot_K$ of v (cf. [9]). But we easily see $:v_{*+}^{-1} * \varpi_{00} \cdot_K$ must diverge, for $v * \varpi_{00} = 0$ in generic ordered expression.

2.2.1 Remarks on ordinary calculus

Next, we note that (1.7) gives in generic ordered expressions that the integrals

$$(u*v)_{*-}^{-1} = -\frac{1}{i\hbar} \int_0^\infty e_*^{s\frac{1}{i\hbar}u*v} ds, \quad (v*u)_{*+}^{-1} = \frac{1}{i\hbar} \int_{-\infty}^0 e_*^{s\frac{1}{i\hbar}v*u} ds$$

exist to give inverses of $u*v$, $v*u$ respectively. Hence the next ones give left/right inverses of u, v

$$(2.14) \quad v^\circ = u*(v*u)_{*+}^{-1}, \quad u^\bullet = v*(u*v)_{*-}^{-1},$$

for it is easy to see that

$$v*v^\circ = 1, \quad v^\circ*v = 1 - \varpi_{00}, \quad u*u^\bullet = 1, \quad u^\bullet*u = 1 - \overline{\varpi}_{00}.$$

The bumping identity $u*f_*(v*u) = f_*(u*v)*u$ gives

$$\begin{aligned} v*(z + \frac{1}{i\hbar}u \circ v)*v^\circ &= z + 1 + \frac{1}{i\hbar}u \circ v, & v^\circ*(z + \frac{1}{i\hbar}u \circ v)*v &= (1 - \varpi_{00})*(z - 1 + \frac{1}{i\hbar}u \circ v). \\ u*(z + \frac{1}{i\hbar}u \circ v)*u^\bullet &= z - 1 + \frac{1}{i\hbar}u \circ v, & u^\bullet*(z + \frac{1}{i\hbar}u \circ v)*u &= (1 - \overline{\varpi}_{00})*(z + 1 + \frac{1}{i\hbar}u \circ v). \end{aligned}$$

The successive use of the bumping identity gives the following useful formula:

$$(2.15) \quad \begin{aligned} (v^\circ)_*^n * \varpi_{00} &= (u*(v*u)_{*+}^{-1})^n * \varpi_{00} = \frac{1}{n!} \left(\frac{1}{i\hbar}u \right)_*^n * \varpi_{00}, \\ (u^\bullet)_*^n * \overline{\varpi}_{00} &= (v*(u*v)_{*-}^{-1})^n * \overline{\varpi}_{00} = \frac{1}{n!} \left(\frac{1}{i\hbar}v \right)_*^n * \overline{\varpi}_{00}. \end{aligned}$$

Next Proposition shows that the vacuum representation involves ordinary elementary calculus.

Proposition 2.6 *In generic ordered expressions, we have*

$$\frac{1}{i\hbar}v*f(u)*\varpi_{00} = f'(u)*\varpi_{00}, \quad v^\circ*f(u)*\varpi_{00} = \frac{1}{i\hbar} \int_{-\infty}^u f(x)dx*\varpi_{00}.$$

That is, $\frac{1}{i\hbar}v*$ represents the differentiation, and its right inverse $v^\circ*$ represents the integration.

Proof Since $v*f(u) = f(u)*v + i\hbar f'(u)$, first one is easy to see. For the second, $[\frac{1}{i\hbar}v*u, u] = u$ and $v*\varpi_{00} = 0$ gives that

$$e_*^{t\frac{1}{i\hbar}v*u} * f(u) * e_*^{-t\frac{1}{i\hbar}v*u} = f(e^t u), \quad e_*^{t\frac{1}{i\hbar}v*u} * \varpi_{00} = e^t * \varpi_{00}.$$

It follows $v^\circ*f(u)*\varpi_{00} = \frac{1}{i\hbar} \int_{-\infty}^0 e^t u * f(e^t u) dt * \varpi_{00}$. Set $x = e^t u$ to obtain the result. \square

By Proposition 2.6, one can begin the whole story with the algebra of ordinary elementary calculus of differentiation and integration. This implies that all strange phenomena mentioned in [9], [10] and [11] are already involved in the ordinary elementary calculus.

2.3 Integrals along closed paths

All closed paths drawn in the Fig.1 give closed paths in the Riemann surface with infinite genus, where dotted lines are in the negative sheets. Let C be a one of such closed path. Then, the integral $\int_C :e_*^{z(\alpha + \frac{1}{i\hbar}u \circ v)} dz:_K$ must satisfy

$$(2.16) \quad :(\alpha + \frac{1}{i\hbar}u \circ v) * \int_C e_*^{z(\alpha + \frac{1}{i\hbar}u \circ v)} dz:_K = : \int_C \frac{d}{dz} e_*^{z(\alpha + \frac{1}{i\hbar}u \circ v)} dz:_K = 0.$$

Residue-like integrals Now, let C be a closed path avoiding the singular points and intersecting the slit even (possibly zero) times. At a first glance, the integral $\int_C e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha)} dz$ looks to relate to the residues of singular points sitting in the inside of C . In fact, this integral is nothing to do with residues. We call this a residue-like integral. The residue-like integral of $e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha)}$ relates to the global topological nature. On the contrary, the residues of $e_*^{z(\alpha + \frac{1}{i\hbar}u \circ v)}$ does not have a global nature.

The bumping identity $u * f_*(v * u) = f_*(u * v) * u$ gives

$$(2.17) \quad u^n * e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha)} = e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha - n)} * u^n, \quad v^n * e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha)} = e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha + n)} * v^n.$$

Hence

$$\int_C :u * e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha)} * v:_K dz = \int_C : (u \circ v - \frac{i\hbar}{2}) * e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha - 1)} :_K dz = (i\hbar)(\frac{1}{2} - \alpha) \int_C :e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha - 1)}:_K dz.$$

Repeat this to obtain

$$\int_C :u^n * e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha)} * v^n:_K dz = (i\hbar)^n (\frac{1}{2} - \alpha)_n \int_C e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha - n)} dz:_K,$$

where $(a)_n = a(a+1) \cdots (a+n-1)$, $(a)_0 = 1$. Similarly we have

$$\begin{aligned} :v^n * \int_C e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha)} dz * u^n:_K &= (i\hbar)^n (-\frac{3}{2} - \alpha)(-\frac{5}{2} - \alpha) \cdots (-\frac{2n+1}{2} - \alpha) \int_C e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha + n)} dz:_K \\ &= (\frac{1}{2} - \alpha)_{-n} \int_C e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha + n)} dz:_K, \end{aligned}$$

by extending the convention $(a)_{-n} = (a-1)(a-2) \cdots (a-n)$. If we use the convention

$$(2.18) \quad \zeta^k = \begin{cases} u^k, & k \geq 0 \\ v^{|k|}, & k < 0, \end{cases} \quad \hat{\zeta}^\ell = \begin{cases} v^\ell, & \ell \geq 0 \\ u^{|\ell|}, & \ell < 0, \end{cases}$$

then we have

$$(2.19) \quad :\zeta^n * \int_C e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha)} dz * \hat{\zeta}^n:_K = (\frac{1}{2} - \alpha)_n \int_C e_*^{z(\frac{1}{i\hbar}u \circ v + \alpha - n)} dz:_K, \quad n \in \mathbb{Z}$$

Hence the essential part of the integral (2.19) is reduced to the case $|\operatorname{Re}(\alpha)| \leq \frac{1}{2}$.

Moreover, recall the associativity

$$(e_*^{s(\alpha + \frac{1}{i\hbar}u \circ v)} * \frac{1}{i\hbar}u \circ v) * e_*^{t(\beta + \frac{1}{i\hbar}u \circ v)} = e_*^{s(\alpha + \frac{1}{i\hbar}u \circ v)} * (\frac{1}{i\hbar}u \circ v * e_*^{t(\beta + \frac{1}{i\hbar}u \circ v)}) = \frac{1}{i\hbar}u \circ v * (e_*^{s(\alpha + \frac{1}{i\hbar}u \circ v)} * e_*^{t(\beta + \frac{1}{i\hbar}u \circ v)})$$

proved by the formal associativity theorem (cf.[11]). By the fundamental product formula (cf.[11]), we see the product remains in the space $\mathbb{C}[u, v]e^{Q(u, v)}$. Since this is continuous w.r.t (s, t) , the integration by $dsdt$ on compact domain $C \times C$ in (2.16) gives that

$$\iint_{C \times C} (e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} * \frac{1}{i\hbar} u \circ v) * e_*^{\zeta(\beta + \frac{1}{i\hbar} u \circ v)} dz d\zeta = \iint_{C \times C} e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} * (\frac{1}{i\hbar} u \circ v * e_*^{\zeta(\beta + \frac{1}{i\hbar} u \circ v)}) dz d\zeta.$$

Since $C \times C$ is compact, the uniform continuity gives

$$\int_C (e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} * \frac{1}{i\hbar} u \circ v) dz * \int_C e_*^{\zeta(\beta + \frac{1}{i\hbar} u \circ v)} d\zeta = \int_C e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} dz * \int_C (\frac{1}{i\hbar} u \circ v * e_*^{\zeta(\beta + \frac{1}{i\hbar} u \circ v)}) d\zeta.$$

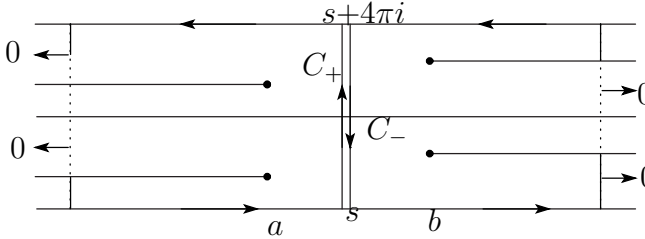
Hence we see

$$\begin{aligned} : -\alpha \int_C e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} dz * \int_C e_*^{\zeta(\beta + \frac{1}{i\hbar} u \circ v)} d\zeta :_K &= : \int_C e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} dz * \frac{1}{i\hbar} u \circ v * \int_C e_*^{\zeta(\beta + \frac{1}{i\hbar} u \circ v)} d\zeta :_K \\ &= : -\beta \int_C e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} dz * \int_C e_*^{\zeta(\beta + \frac{1}{i\hbar} u \circ v)} d\zeta :_K. \end{aligned}$$

Thus, if $\alpha \neq \beta$, then

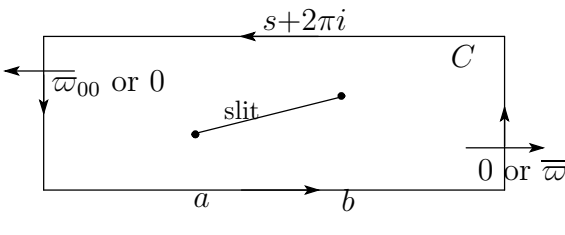
$$(2.20) \quad : \int_C e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} dz * \int_C e_*^{\zeta(\beta + \frac{1}{i\hbar} u \circ v)} d\zeta :_K = 0 \quad (\alpha \neq \beta).$$

In the case $\alpha = 0$ in generic K , we see easily



$$\begin{aligned} : \int_{C_+} e_*^{z \frac{1}{i\hbar} u \circ v} dz :_K &= 4\pi : \varpi_*(s) :_K \\ : \int_{C_-} e_*^{z \frac{1}{i\hbar} u \circ v} dz :_K &= -4\pi : \varpi_*(s) :_K \\ \text{for } \lim_{\tau \rightarrow \pm\infty} e_*^{(\tau+it) \frac{1}{i\hbar} u \circ v} &= 0. \end{aligned}$$

On the other hand, as $\lim_{\tau \rightarrow -\infty} e_*^{(\tau+it) \frac{1}{i\hbar} u \circ v} = \varpi_{00}$, $\lim_{\tau \rightarrow \infty} e_*^{(\tau+it) \frac{1}{i\hbar} u \circ v} = 0$, we have

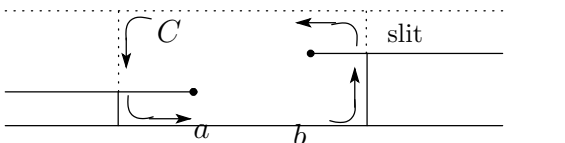


$$: \int_C e_*^{z \frac{1}{i\hbar} u \circ v} dz :_K = -2\pi : \varpi_{00} :_K.$$

Similarly, we have

$$: \int_C e_*^{z \frac{1}{i\hbar} v \circ u} dz :_K = 2\pi : \overline{\varpi}_{00} :_K.$$

The next one is given by $2\pi i$ -alternating periodicity and by Cauchy's integral theorem:



$$: \int_C e_*^{z \frac{1}{i\hbar} u \circ v} dz :_K = 2 \int_{-\infty}^{\infty} e_*^{s \frac{1}{i\hbar} u \circ v} ds :_K = 2 : \delta_*(\frac{1}{\hbar} u \circ v) :_K.$$

2.3.1 Residues and secondary residues

Let $\hat{a}+i\sigma_{\hat{a}}, \hat{b}+i\sigma_{\hat{b}}$ be the singular points such that $0 < \sigma_{\hat{a}} < 2\pi, 0 < \sigma_{\hat{b}} < 2\pi$.

Now, let σ be the representative of the isolated singular points $s_a+2\pi il$ or $s_b+2\pi il$. We make a double covering space (the Riemann surface) of a neighborhood D of σ to make a single-valued holomorphic function, and then we take its Laurent expansion.

Recall generic K -ordered expression of $*$ -exponential function $e_*^{z(\frac{1}{i\hbar}u\circ v+\alpha)}$ is written in the form

$$:e_*^{z(\frac{1}{i\hbar}u\circ v+\alpha)}:_K = \frac{e^{\alpha z}}{\sqrt{g(z)}} e^{\frac{1}{i\hbar}H(z,u,v)}.$$

By the first comment in §1.1, $\sigma, g(z), H(z,u,v)$ are given on a neighborhood of an isolated singular point by using a holomorphic functions $h(z), a(z,u,v), b(z,u,v)$ as

$$\begin{aligned} g(z) &= (z-\sigma)h(z), \quad h(\sigma) \neq 0, \\ H(z,u,v) &= \frac{a(z,u,v)}{z-\sigma} + b(z,u,v), \quad a(\sigma,u,v) \neq 0, \quad H(z,0,0) = 0. \end{aligned}$$

Hence, setting $z = \sigma + s^2$, the Laurent series of $\frac{1}{\sqrt{g(z)}}$ at $s = 0$ is given by

$$\frac{1}{\sqrt{g(z)}} = \frac{1}{s}(h_0 + h_1 s^2 + h_2 s^4 + \dots), \quad h_0 \neq 0,$$

without terms of even degree. If $h_k = 0$ for all $k \geq 1$, then there is only one singular point $s = 0$. Hence, if $g(z)$ has many zeros, then $\frac{1}{\sqrt{g(z)}}$ has terms of positive degree. The Laurent series of $:e_*^{z\frac{1}{i\hbar}u\circ v}:_K$ at the singular point σ is written as

$$\begin{aligned} (2.21) \quad & :e_*^{(\sigma+s^2)(\frac{1}{i\hbar}u\circ v+\alpha)}:_K = \\ & \dots + \frac{a_{-(2k+1)}(\sigma, K)}{s^{2k+1}} + \dots + \frac{a_{-1}(\sigma, K)}{s} + a_1(\sigma, K)s + \dots + a_{2k+1}(\sigma, K)s^{2k+1} + \dots \end{aligned}$$

without terms of even degree, where

$$\begin{aligned} (2.22) \quad & \frac{1}{2\pi i} \int_{\tilde{C}} s^{-2k} :e_*^{(\sigma+s^2)(\frac{1}{i\hbar}u\circ v+\alpha)}:_K ds = \text{Res}_{s=0} (s^{-2k} e_*^{(\sigma+s^2)(\frac{1}{i\hbar}u\circ v+\alpha)}:_K) = a_{2k-1}(\sigma, K) \\ & \frac{1}{2\pi i} \int_{\tilde{C}} s^{-2k-1} :e_*^{(\sigma+s^2)(\frac{1}{i\hbar}u\circ v+\alpha)}:_K ds = \text{Res}_{s=0} (s^{-2k-1} e_*^{(\sigma+s^2)(\frac{1}{i\hbar}u\circ v+\alpha)}:_K) = 0. \end{aligned}$$

by setting \tilde{C} a small circle with the center at 0. As it is mentioned above, there is nontrivial positive terms of even degree, as $:e_*^{z\frac{1}{i\hbar}u\circ v}:_K$ has infinitely many singular points.

The next one the fundamental property of residues given by the integration by parts:

Proposition 2.7 *In a generic ordered expression K which is fixed, the Laurent coefficients $a_{(2k-1)}(\sigma, K)$ of $:e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u\circ v)}:_K$ satisfy*

$$\begin{aligned} (2.23) \quad & 2\pi i :(\alpha + \frac{1}{i\hbar}u\circ v):_K *_K a_{2k-1}(\sigma, K) = \int_{\tilde{C}} \frac{1}{2} s^{-2k-1} : \frac{d}{ds} e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u\circ v)}:_K ds \\ & = -\frac{2k+1}{2} \int_{\tilde{C}} s^{-2(k+1)} :e_*^{(\sigma+s^2)\frac{1}{i\hbar}u\circ v}:_K ds = -2\pi i \frac{2k+1}{2} a_{2k+1}(\sigma, K). \end{aligned}$$

Calculation on \pm -sheets

Although we use only the case $p = 2$, it is useful to fix a method to compute the residue at a p -branching ($\sqrt[p]{z}$ -) singular point by using p -sheets together with $\omega = e^{\frac{2\pi i}{p}}$.

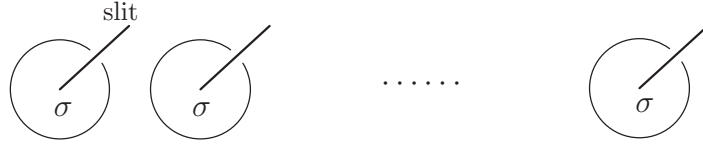
To treat $\frac{1}{\sqrt[p]{z}}$ as a single valued function, we take p -covering $z = s^p$ and regard $\frac{1}{\sqrt[p]{z}}$ as

$$\frac{1}{\sqrt[p]{z}} = \left\{ \frac{1}{s}, \frac{1}{\omega s}, \dots, \frac{1}{\omega^{p-1}s} \right\}$$

preparing p -sheets, and

$$\frac{1}{\sqrt[p]{z}} d\sqrt[p]{z} = \left\{ \frac{1}{s} ds, \frac{1}{\omega s} d(\omega s), \dots, \frac{1}{\omega^{p-1}s} d(\omega^{p-1}s) \right\}.$$

Hence computing the contour $\int_{\tilde{C}} \frac{1}{s} ds$ is nothing but the calculation $\int_C \frac{1}{\sqrt[p]{z}} d\sqrt[p]{z}$ on each



sheet where the existence of slit keeps the integrand single value. But as a result, the integral equals formally $p \int_C \frac{1}{pz} dz = \int_C \frac{1}{z} dz$.

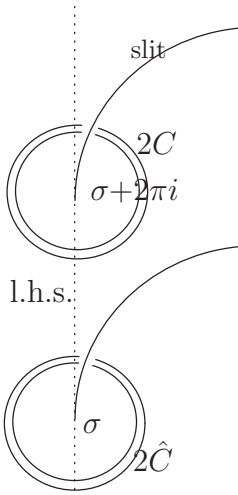
In the case of $e_*^{z \frac{1}{ih} u \circ v}$, we have only to use two sheets, and we can apply this method to treat several singular points at the same time. The next one gives the periodical properties of residues of $*$ -exponential functions: Let $\text{Res}(\sigma + 2\pi i k, \alpha, K)$ be the residue of $e_*^{z(\frac{1}{ih} u \circ v + \alpha)}$ at $z = \sigma + 2\pi i k$.

Lemma 2.2 *If α is an integer or a half-integer, then in generic ordered expression, the residue has the alternating $2\pi i$ -periodicity*

$$\text{Res}(\sigma + 2\pi i k, \alpha, K) = (-1)^k \text{Res}(\sigma, \alpha, K).$$

Proof is given by showing $\text{Res}(\sigma, \alpha, K) + \text{Res}(\sigma + 2\pi i, \alpha, K) = \frac{1}{2}(\text{Res}(\sigma, \alpha, K) + \text{Res}(\sigma + 2\pi i, \alpha, K))$.

The l.h.s.(resp.r.h.s.) of the vertical dotted line is the region that $e_*^{z \frac{1}{ih} u \circ v}$ is $2\pi i$ -periodic (resp. alternating $2\pi i$ -periodic). Double circles are the paths of integrals



$$\int_{C^2} e_*^{z(\alpha + \frac{1}{ih} u \circ v)} d\sqrt{z - \sigma}, \quad \int_{\hat{C}^2} e_*^{z(\alpha + \frac{1}{ih} u \circ v)} d\sqrt{z - \sigma - 2\pi i}$$

in each sheet. But in fact these are

$$(2.24) \quad 2 \int_C e_*^{z(\alpha + \frac{1}{ih} u \circ v)} d\sqrt{z - \sigma}, \quad 2 \int_{\hat{C}} e_*^{z(\alpha + \frac{1}{ih} u \circ v)} d\sqrt{z - \sigma - 2\pi i}.$$

Sum up all integrals to obtain

$$2\pi i \text{Res}(\sigma, \alpha, K) + 2\pi i \text{Res}(\sigma + 2\pi i, \alpha, K),$$

but the alternating $2\pi i$ -periodicity of the integrand on the r.h.s. or on the l.h.s. erases the half of the quantity. \square

Remark here by minding about slits that the integrals (2.24) may be replaced by

$$\int_C e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} \frac{dz}{\sqrt{z - \sigma}}, \quad \int_{\hat{C}} e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} \frac{dz}{\sqrt{z - \sigma - 2\pi i}}.$$

In this way of expressing of integrals, we can consider a wider region beyond neighborhoods of singular points. Fix a singular point σ and consider a contour integral

$$(2.25) \quad \int_{C^2} e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} d\sqrt{z - \sigma} = \int_{C^2} e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} \frac{dz}{2\sqrt{z - \sigma}}$$

where C is a closed path in \mathbb{C} , which may include other singular points inside, and C^2 is the same path rounding twice on the same path so that the integrand to be closed.

Next Theorem together with Cauchy integral theorem shows that the integral (2.25) depends only to the prescribed singular point σ :

Theorem 2.2 *If a singular point σ is fixed, then the integral*

$$(2.26) \quad \frac{1}{2\pi i} \int_{\tilde{C}^2} e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} d\sqrt{z - \sigma}$$

over a small circle \tilde{C} centered at an arbitrary point is given by

$$\frac{1}{2\pi i} \int_{\tilde{C}^2} e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} d\sqrt{z - \sigma} = \begin{cases} \text{Res}(\sigma, \alpha, K) & \text{if } \sigma \text{ is the inside of } \tilde{C} \\ \frac{1}{2} \text{Res}(\sigma, \alpha, K) & \text{if } \sigma \in \tilde{C} \\ 0 & \text{if } \sigma \text{ is the outside of } \tilde{C} \end{cases}$$

On the other hand, if σ is not a singular point, then the integral (2.26) vanishes everywhere.

Proof The integral vanishes when there is no singular point inside. Let σ' be another singular point inside \tilde{C} . In a neighborhood of σ' , the Laurent series of $:e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)}:_K$ is given in the form

$$\cdots + \frac{a_{-3}}{\sqrt{z - \sigma'}^3} + \frac{a_{-1}}{\sqrt{z - \sigma'}} + a_1 \sqrt{z - \sigma'} + \cdots$$

without terms of even degree w.r.t. $\sqrt{z - \sigma'}$. By setting $s^2 = z - \sigma'$, we have $\sqrt{z - \sigma} = \sqrt{s^2 + \sigma' - \sigma}$ and the integral we have to consider is

$$\cdots + \int_{\tilde{C}} \frac{a_{-3}}{s^3} \frac{s ds}{\sqrt{s^2 + \sigma' - \sigma}} + \int_{\tilde{C}} \frac{a_{-1}}{s} \frac{s ds}{\sqrt{s^2 + \sigma' - \sigma}} + \int_{\tilde{C}} a_1 s \frac{s ds}{\sqrt{s^2 + \sigma' - \sigma}} + \cdots = 0$$

for the Taylor series of $\frac{1}{\sqrt{s^2 + \sigma' - \sigma}}$ has no term of odd degree. The second statement is proved by applying the argument above to the case that σ' is a regular point. \square

It is remarkable that this theorem makes us possible to concern with individual/subjective singular point depending on the individual expression parameters.

2.3.2 Residues and Laurent coefficients

Recall the formula (1.7)

$$:e_*^{t\frac{1}{i\hbar}2u\circ v}:_K = \frac{2}{\sqrt{\Delta^2 - (e^t - e^{-t})^2 \delta \delta'}} e^{\frac{1}{i\hbar} \frac{e^t - e^{-t}}{\Delta^2 - (e^t - e^{-t})^2 \delta \delta'} ((e^t - e^{-t})(\delta u^2 + \delta' v^2) + 2\Delta uv)},$$

where $\Delta = e^t + e^{-t} - c(e^t - e^{-t})$. Using this, we compute the Laurent coefficients $a_{2k-1}(\sigma, K)$, where $K = \begin{bmatrix} \delta & c \\ c & \delta' \end{bmatrix}$ and σ is a singular point.

Now, we set

$$s^2 = e^t + e^{-t} - (c + \sqrt{\delta \delta'})(e^t - e^{-t}), \quad \hat{s}^2 = e^t + e^{-t} - (c - \sqrt{\delta \delta'})(e^t - e^{-t})$$

so that the singular points are obtained by $s^2 = 0$ or $\hat{s}^2 = 0$. Since $\delta \delta' \neq 0$ is assumed in general, singular points satisfy $s^2 = 0$ or $\hat{s}^2 = 0$ exclusively. One may set $\hat{s}^2 = \hat{s}^2(s)$, $\hat{s}^2(0) \neq 0$, and similarly $s^2 = s^2(\hat{s})$, $s^2(0) \neq 0$.

Hence,

$$\hat{s}^2 - s^2 = 2\sqrt{\delta \delta'}(e^t - e^{-t}), \quad \hat{s}^2 + s^2 = 2(e^t + e^{-t}) - c \frac{\hat{s}^2 - s^2}{\sqrt{\delta \delta'}}.$$

It follows that $2\Delta = \hat{s}^2 + s^2$. Plugging these we have

$$(2.27) \quad :e_*^{t\frac{1}{i\hbar}2u\circ v}:_K = \frac{2}{s\hat{s}} \exp\left(\frac{1}{4i\hbar} \left(\frac{\hat{s}^2}{s^2} - \frac{s^2}{\hat{s}^2}\right) \left(\frac{u}{\sqrt{\delta'}} + \frac{v}{\sqrt{\delta}}\right)^2\right) \exp\left(-\frac{1}{2i\hbar} \left(\left(\frac{u}{\sqrt{\delta'}}\right)^2 + \left(\frac{v}{\sqrt{\delta}}\right)^2\right)\right).$$

Note that the variable s (resp. \hat{s}) may be viewed as the local coordinate around the singular point given by $s = 0$ (resp. $\hat{s} = 0$). It follows that

$$\text{Res}_{s=0}:e_*^{(\sigma+s^2)\frac{1}{i\hbar}2u\circ v}:_K = \frac{1}{2\pi i} e^{-\frac{1}{2i\hbar} \left(\left(\frac{u}{\sqrt{\delta'}}\right)^2 + \left(\frac{v}{\sqrt{\delta}}\right)^2\right)} \frac{2}{\hat{s}(0)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!k!} \left(\frac{1}{4i\hbar} \left(\frac{u}{\sqrt{\delta'}} + \frac{v}{\sqrt{\delta}}\right)\right)^{2k}.$$

$\text{Res}_{\hat{s}=0}:e_*^{(\sigma+\hat{s}^2)\frac{1}{i\hbar}2u\circ v}:_K$ is obtained by replacing $\hat{s}(0)$ by $s(0)$. Explicit formula of Laurent coefficients is not easy to write down, but it is clear that there is no term of even degree.

Secondary residues do not appear If there is a nontrivial term $\frac{1}{s^2}$ in the Laurent series of $f_*(\sigma + s^2, u, v)$ at a singular point σ , then the residue-like integral gives $2\pi i a_{-2}$.

$$(2.28) \quad a_{-2}(\sigma, K) = \frac{1}{4\pi i} \int_{C^2} :f_*(\sigma + s^2, u, v):_K ds^2$$

will be called the **secondary residue** at σ , where C^2 is a small circle with center at σ round twice. We denote this by $\text{Res}_2(\sigma, \alpha, K)$. The secondary residues are much easier to calculate as we have only to use double loops, but the secondary residue does not appear in our case $e_*^{z(\alpha + \frac{1}{i\hbar}u*v)}$. These observation may be summarized as follows:

Theorem 2.3 *If a singular point σ is fixed, then the integral $\frac{1}{2\pi i} \int_{C^2} e_*^{z(\alpha + \frac{1}{i\hbar} u \circ v)} dz$ on a small circle centered at σ vanishes always.*

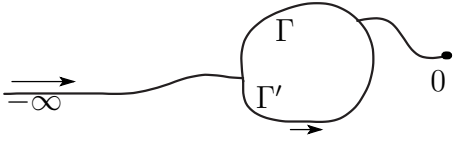
Another word, the secondary residue vanishes identically at any isolated $\sqrt{}$ branching singular point.

To define the such an integral at a singular point σ , one has to use $d(s^2)$ instead of ds in the definition (2.22) of $a_k(K)$, that is,

$$\int_{\tilde{C}} s^{k-1} :e_*^{(\sigma+s^2)\frac{1}{i\hbar} u \circ v} :_K d(s^2) = 2 \int_{\tilde{C}} s^k :e_*^{(\sigma+s^2)\frac{1}{i\hbar} u \circ v} :_K ds = 2a_{-(k+1)},$$

but the integral may be simply written as $\int_{C^2} : \sqrt{z-\sigma}^{k-1} e_*^{z\frac{1}{i\hbar} u \circ v} :_K dz$. For $k = 1$, this integral is much easier to treat. But at the singular point σ , this must give $4\pi i a_{-2}(K)$ and (2.21) shows there is no term of even degree.

Proposition 2.8 *The integral $\int_C :e_*^{t(\frac{1}{i\hbar} u \circ v + \alpha)} :_K dt$ is obtained as the difference of two different inverses of $\frac{1}{i\hbar} u \circ v + \alpha$. In contrast, the residue $\text{Res}(\sigma, \alpha)$ can not be given by such a way.*



Proof Suppose $\text{Re} \alpha > 0$. Then the both integrals $\int_{-\infty}^0 :e_*^{t(\frac{1}{i\hbar} u \circ v + \alpha)} :_K dt$ along the path Γ and Γ' converge to give inverses of $\frac{1}{i\hbar} u \circ v + \alpha$. The difference is the integral over C . If $\alpha < 1$, then the integral $-\int_0^\infty :e_*^{t(\frac{1}{i\hbar} u \circ v + \alpha)} :_K dt$ also gives an inverse. Suppose a residue $a_{-1}(\alpha) = \text{Res}(\sigma, \alpha)$ is written as a difference

$$a_{-1}(\alpha) = (\frac{1}{i\hbar} u \circ v + \alpha)_{*\mu}^{-1} - (\frac{1}{i\hbar} u \circ v + \alpha)_{*\nu}^{-1}.$$

Then, we must have $a_1(\alpha) = (\frac{1}{i\hbar} u \circ v + \alpha) * a_{-1}(\alpha) = 0$, but $a_1(\alpha) \neq 0$ for generic α . \square

By vanishing of $a_{-2}(K)$, we have no need to care about residue-like quantity in the computation of principal values:

Proposition 2.9 *Let $I_\circ(K) = (a, b)$ be the exchanging interval. In generic ordered expressions,*

$$\text{vp-} \int_0^{2\pi} :e_*^{(a+i\tau)\frac{1}{i\hbar} u * v} :_K d\tau = 2\pi \varpi_{00}$$

In particular, $\text{vp-} \int_0^{2\pi} :e_^{(a+i\tau)\frac{1}{i\hbar} u * v} :_K d\tau$ satisfies the differential equation $(\frac{1}{i\hbar} u * v) * f = 0$.*

Moreover, if ℓ is an integer, then the exponential law of the integrand shows

$$\text{vp-} \int_0^{2\pi} :e_*^{(a+i\tau)(\frac{1}{i\hbar} u * v + \ell)} :_K d\tau = \begin{cases} 0 & \ell > 0 \\ 2\pi \varpi_{00} & \ell = 0 \\ \infty & \ell < 0. \end{cases}$$

Similarly, taking the growth order into account, we have

$$\text{vp-} \int_0^{2\pi} :e_*^{(b+i\tau)(\frac{1}{i\hbar} u * v + \ell)} :_K d\tau = \begin{cases} 0 & \ell < 1 \\ 2\pi \overline{\varpi}_{00} & \ell = 1 \\ \infty & \ell > 1. \end{cases}$$

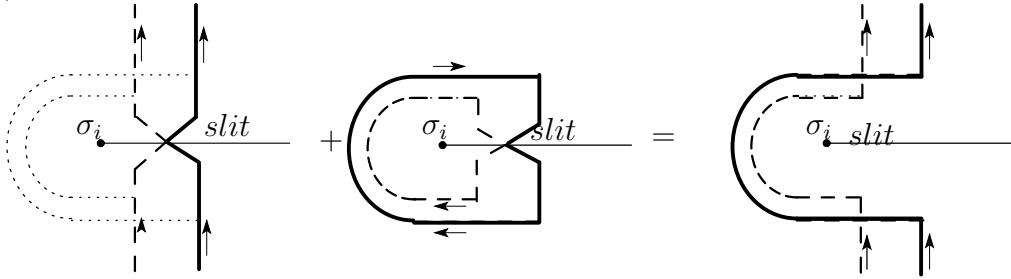
Let σ be a moving singular point which we want to concern. Theorem 2.4 below shows that the residues of $:e_*^{z(\alpha+\frac{1}{i\hbar}u\circ v)}:_K$ at σ can be computed by the integral along a fixed big closed path.

Theorem 2.4 *Even though other singular points may lie inside C , the integral*

$\int_{C^2} :e_^{z(\alpha+\frac{1}{i\hbar}u\circ v)}:_K d\sqrt{z-\sigma}$ on the closed path C^2 gives $\ell \times \text{Res}(\sigma, \alpha, K)$, where ℓ is the winding number of C around σ .*

On the other hand, if σ is not a singular point, then the integral $\int_{C^2} :e_^{z(\alpha+\frac{1}{i\hbar}u\circ v)}:_K d\sqrt{z-\sigma}$ on the closed path C^2 vanishes.*

Proof It is obvious by the Cauchy integral theorem if C does not involve other singular point inside. Suppose that C is a simple closed curve. Let $\sigma_i, i = 0, 1, \dots, n$ be singular points inside C , and $\sigma = \sigma_0$. The proof is given by showing that C^2 is homologically equivalent to $C_0^2 \cup C_1^2 \cup \dots \cup C_n^2$. This is trivial if there is no slit. But, Theorem 2.2 and the next figure for $\sigma_i, i \neq 0$, show that the same proof is valid. This proves also the second statement.



Homological nature in the double cover

□

3 The differential equation $\frac{\nabla}{d\sigma} f_\sigma(u, v) = :(\frac{1}{i\hbar}u\circ v + \alpha):_K *_K f_\sigma(u, v)$

In ordinary complex calculus, residues mainly relate to the global nature of Riemann surfaces. In contrast, $\int_{C^2} :e_*^{z(\alpha+\frac{1}{i\hbar}u\circ v)}:_K d\sqrt{z-\sigma}$ appears to depend on the position σ which moves by the expression parameter K . We see indeed the following:

Theorem 3.1 *Suppose C^2 is fixed and the singular point σ moves without crossing C , or C is an infinitesimally small circle with center at σ , and moving together with σ . Then*

$$\frac{d}{dt}\bigg|_{t=0} \int_{C^2} :e_*^{z(\alpha+\frac{1}{i\hbar}u\circ v)}:_K d\sqrt{z-\sigma-t} = :(\alpha+\frac{1}{i\hbar}u\circ v):_K *_K \int_{C^2} :e_*^{z(\alpha+\frac{1}{i\hbar}u\circ v)}:_K d\sqrt{z-\sigma}.$$

Suppose C is any smooth closed curve in the complex plane \mathbb{C} avoiding singular points. To consider the movement of a singular point σ , we use the integral

$$R_{-2k-1}(z, u, v; K) = \int_{C^2} :s^{2k+2} e_*^{(z+s^2)(\alpha+\frac{1}{i\hbar}u\circ v)}:_K ds, \quad k \in \mathbb{Z}.$$

If z is an independent variable, then integration by parts gives that this must satisfy

$$(3.1) \quad \partial_z R_{-2k-1}(z, u, v; K) = :(\alpha+\frac{1}{i\hbar}u\circ v):_K *_K R_{-2k-1}(z, u, v; K).$$

However, Theorems 2.2, 2.4 show that the integral

$$\int_{C^2} :s^{2k+2} e_*^{(z+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)} :_K ds$$

vanishes for all regular points $z \notin \Sigma_K$. $R_{-2k-1}(z, u, v; K)$ gives a nontrivial value only when $z \in \Sigma_K$. That is, z cannot be an independent variable in $R_{-2k-1}(z, u, v; K)$. Summarizing these observation we have the following by using (2.22):

Theorem 3.2 *For every Laurent polynomial $\psi(s^2, s^{-2})$, the integral*

$$\Phi[\psi](z, K, u, v) = \int_{C^2} :\psi(s^2, s^{-2}) e_*^{(z+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)} :_K ds$$

is supported only on the set $\{(z, K); z \in \Sigma_K\}$, where C is any smooth closed curve in the complex plane \mathbb{C} avoiding singular points.

Let \mathfrak{S} be the totality of K such that Σ_K is a set of simple singularities. Then, $\mathcal{S} = \{(z, K); z \in \Sigma_K, K \in \mathfrak{S}\}$ is a holomorphic submanifold of codimension 1. Note that $\Phi[\psi](z, K, *, *)$ is a holomorphic on the submanifold \mathcal{S} .

Co-moving differentials Let σ be a variable $\sigma \in \mathbb{C}$ together with an expression parameter $K(\sigma)$ such that $\sigma \in \Sigma_{K(\sigma)}$. We define

$$(3.2) \quad \frac{\nabla}{d\sigma} f(\sigma, K(\sigma); u, v) = \partial_\sigma f(\sigma, K; u, v)|_{K=K(\sigma)} = \partial_z(f(\sigma, K(\sigma); u, v)) - \frac{i\hbar}{4} \dot{K}(\sigma)(f)$$

where $\frac{i\hbar}{4} \dot{K}(\sigma)(f)$ is the infinitesimal intertwiner (cf.(1.4)) given by

$$(3.3) \quad \frac{i\hbar}{4} \dot{K}(\sigma)(f) = \frac{i\hbar}{4} \sum_{ij} \frac{dK^{ij}(\sigma)}{d\sigma} \partial_{u^i} \partial_{u^j} f, \quad (u, v) = (u^1, u^2)$$

for every $Hol(\mathbb{C}^2)$ -valued function $f(\sigma, K; *, *)$. Intuitively, this may be written as

$$(3.4) \quad f(\sigma+\delta, K(\sigma+\delta); u, v) = I_{K(\sigma)}^{K(\sigma+\delta)} \left(:e_*^{\delta(z+\frac{1}{i\hbar}u \circ v)} :_{K(\sigma)} *_{K(\sigma)} f(\sigma, K(\sigma); u, v) \right),$$

where δ is an infinitesimal. $\Phi[\psi](\sigma, K, u, v)$ given in Theorem 3.2 satisfies

$$\frac{\nabla}{d\sigma} \Phi[\psi](\sigma, K, u, v) = :(\alpha + \frac{1}{i\hbar} u \circ v) :_{K(\sigma)} *_{K(\sigma)} \Phi[\psi](\sigma, K, u, v).$$

This is proved as follows:

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{1}{t} \left(I_{K(\sigma+t)}^{K(\sigma)} \int_{C^2} :\psi(s, s^{-1}) e_*^{(\sigma+t+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)} :_{K(\sigma+t)} ds - \int_{C^2} :\psi(s, s^{-1}) e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)} :_{K(\sigma)} ds \right) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_{C^2} :\psi(s, s^{-1}) e_*^{(\sigma+t+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)} :_{K(\sigma)} ds - \int_{C^2} \psi(s, s^{-1}) :e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)} :_{K(\sigma)} ds \right) \\ &= \int_{C^2} \psi(s, s^{-1}) \lim_{t \rightarrow 0} \frac{1}{t} \left(:e_*^{(\sigma+t+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)} - e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)} :_{K(\sigma)} \right) ds \\ &= :(\alpha + \frac{1}{i\hbar} u \circ v) * \int_{C^2} \psi(s, s^{-1}) e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)} ds :_{K(\sigma)}. \end{aligned}$$

Since every Laurent coefficients of $:e_*^{\delta(z+\frac{1}{i\hbar}u\circ v)}:_{K(\sigma)}$ is obtained by this integral, we see in particular, for every σ , the coefficient $a_{2k-1}(\sigma, K(\sigma))$ of Laurent series of $:e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u\circ v)}:_{K(\sigma)}$ at a singular point σ satisfies the equation

$$(3.5) \quad \frac{\nabla}{d\sigma} f(\sigma, K(\sigma); u, v) = :(\alpha + \frac{1}{i\hbar}u\circ v):_K *_K f(z, K(\sigma); u, v).$$

Star-product integrals Viewing (3.5) as a differential equation, we can make the solution by the product integrals, if one can expect the convergence.

Let $\sigma(t)$ be a piecewise smooth curve in \mathbb{C} for $0 \leq t \leq 1$. We assume that the expression parameter K moves together with σ and we suppose $\sigma(t) \in \Sigma_{K(\sigma(t))}$. For a division

$$\Delta; \quad 0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$$

of $[0, 1]$, we first define the product integral $\mathcal{P}_\Delta(\sigma(t))$ inductively by setting

$$\mathcal{P}_\Delta(\sigma(0)) = f(\sigma(0), K_{\sigma(0)}, u, v)$$

and

$$\mathcal{P}_\Delta(\sigma(t)) = I_{K(\sigma_i)}^{K(\sigma(t))} \left((:e_*^{(\sigma(t)-\sigma(t_i))(\alpha+\frac{1}{i\hbar}u\circ v)}:_{K(\sigma(t_i))}) *_K \mathcal{P}_\Delta(\sigma(t_i)) \right), \quad \sigma_i < t \leq \sigma(t_{i+1})$$

where $I_{K(\sigma)}^{K(\sigma')}$ is the intertwiner defined by (1.2) (cf. also (1.4)). Note that $\mathcal{P}_\Delta(\sigma(t))$ is computed under a $K(\sigma(t))$ -expression.

We say that the product integral converges, if by setting $|\Delta| = \max\{|t_i - t_{i-1}|\}$ the limit $\mathcal{P}_d(\sigma(t)) = \lim_{|\Delta| \rightarrow 0} \mathcal{P}_\Delta(\sigma(t))$ exists. It is not hard that if the product integral converges then it satisfies

$$\frac{d}{dt} \mathcal{P}_d(\sigma(t)) = \frac{d\sigma}{dt}(t) :(\alpha + \frac{1}{i\hbar}u\circ v):_{K(\sigma(t))} *_K \mathcal{P}_d(\sigma(t)), \quad \mathcal{P}_d(0) = f(\sigma(0), K_{\sigma(0)}, u, v).$$

Note that $\frac{d}{dt} f(z(t)) = \frac{df}{dz}(z(t)) \frac{dz(t)}{dt}$ for every holomorphic function $f(z)$. The above identity shows that $\frac{d\mathcal{P}_d(\sigma)}{d\sigma}$ is well-defined and

$$(3.6) \quad \frac{d\mathcal{P}_d}{d\sigma}(\sigma(t)) = :(\alpha + \frac{1}{i\hbar}u\circ v):_{K(\sigma(t))} *_K \mathcal{P}_d(\sigma(t)), \quad \mathcal{P}_d(\sigma(0)) = f(\sigma(0), K_{\sigma(0)}, u, v).$$

It is remarkable that even if the starting point $\sigma(0)$ and the ending point $\sigma(1)$ are fixed, the product integral may depend on the path $\sigma(t)$ from $\sigma(0)$ to $\sigma(1)$. There is no general rule to select a specific path from $\sigma(0)$ to $\sigma(1)$. The natural variational problem degenerates.

Note that $\frac{d}{dt} \mathcal{P}_d(\sigma(t))$ is different from an ordinary differentiation. This is given as

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \left(I_{K(\sigma+\delta)}^{K(\sigma)} \mathcal{P}_d(\sigma(t+\delta)) - \mathcal{P}_d(\sigma(t)) \right).$$

Hence it is better to write (3.6) in the form

$$(3.7) \quad \frac{\nabla}{d\sigma} \mathcal{P}_d(\sigma(t)) = :(\alpha + \frac{1}{i\hbar}u\circ v):_{K(\sigma(t))} *_K \mathcal{P}_d(\sigma(t)), \quad \mathcal{P}_d(\sigma(0)) = f(\sigma(0), K_{\sigma(0)}, u, v).$$

The next one is fundamental

Proposition 3.1 For every $s \neq 0$, then $:e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)}:_{K(\sigma)}$ satisfies

$$\frac{\nabla}{d\sigma} :e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)}:_{K(\sigma)} = :(\alpha+\frac{1}{i\hbar}u \circ v):_{K(\sigma)} *_{K(\sigma)} :e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)}:_{K(\sigma)}.$$

Proof This is computed as follows:

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{1}{t} \left(I_{K(\sigma+t)}^{K(\sigma)} :e_*^{(\sigma+t+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)}:_{K(\sigma+t)} - :e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)}:_{K(\sigma)} \right) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left(:e_*^{(\sigma+t+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)}:_{K(\sigma)} - :e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)}:_{K(\sigma)} \right) \\ &= :(\alpha+\frac{1}{i\hbar}u \circ v):_{K(\sigma)} *_{K(\sigma)} :e_*^{(\sigma+s^2)(\alpha+\frac{1}{i\hbar}u \circ v)}:_{K(\sigma)}. \end{aligned}$$

Parallel functions Viewing ∇ the notion of co-moving derivative, we extend (3.2) as the covariant/comoving differentiation not only for $f(\sigma, K(\sigma); u, v)$, but also for functions $f(\sigma, K(\sigma))$ without u, v by

$$\nabla_\sigma f(\sigma, K(\sigma)) = \partial_\sigma f(\sigma, K)|_{K=K(\sigma)}.$$

Let $(a_{ij}(\sigma))$ be a matrix such that $\sum_{i,j} a_{ij}(\sigma) K^{ij}(\sigma) = 0$. Then $f(\sigma, K) = \int^\sigma \sum_{i,j} a_{ij}(\tau) d\tau K^{ij}$ is a parallel function.

Given $K(\sigma)$, parallel functions forms a commutative algebra. We call these *parallel functions on* $K = K(\sigma)$ and denote this by $\mathcal{P}[K(\sigma)]$.

3.1 Co-moving expression parameters

Now consider a general expression parameter $K = \begin{bmatrix} \delta & c \\ c & \delta' \end{bmatrix}$, and set $u \circ v = \frac{1}{2}(u * v + v * u)$.

In this section we use the variable t instead of σ , and we think of the expression parameter K as moving together with the parameter t , indicating a specified singular point. Given t , we choose $K(t)$ so that $t \in \Sigma_{K(t)}$, but different from the case of one variable, we have a lot of choices of $K(t)$. Recall that $\Sigma_{K(t)}$ is the singular set of $:e_*^{z\frac{1}{i\hbar}u \circ v}:_{K(t)}$. In what follows, we think of δ, δ' and c as functions of t . Under this notation, the infinitesimal intertwiner is given by

$$(3.8) \quad \frac{i\hbar}{4} \dot{K} = \frac{i\hbar}{4} (\dot{\delta}(t) \partial_u^2 + 2\dot{c}(t) \partial_u \partial_v + \dot{\delta}'(t) \partial_v^2),$$

where $\dot{a} = \frac{d}{dt} a(t)$.

As $t \in \Sigma_{K(t)}$, t must satisfy

$$(3.9) \quad (e^{t/2} + e^{-t/2} - c(t)(e^{t/2} - e^{-t/2}))^2 - (e^{t/2} - e^{-t/2})^2 \delta(t) \delta'(t) = 0$$

by (1.7). By this $c(t), \delta(t) \delta'(t)$ must be singular at $t = 0$ and we have

$$(3.10) \quad c(t) = \pm \sqrt{\delta(t) \delta'(t)} + \frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}}, \quad t \neq 0.$$

On the other hand, noting that $:\frac{1}{i\hbar}u \circ v:_K = \frac{1}{i\hbar}uv + \frac{1}{2}c$, the K -ordered product of $:(\frac{1}{i\hbar}u \circ v + \alpha):_K *_K f(u, v)$ is written precisely as

$$(3.11) \quad \begin{aligned} & \left(\frac{c}{2} + \alpha + \frac{1}{i\hbar}uv\right)f(u, v) + ((c+1)u + \delta v)\partial_u f(u, v) + (\delta' u + (c-1)v)\partial_v f(u, v) \\ & + \frac{i\hbar}{4}(\delta(c+1)\partial_u^2 f(u, v) + (\delta\delta' + c^2 - 1)\partial_u \partial_v f(u, v) + \delta'(c-1)\partial_v^2 f(u, v)). \end{aligned}$$

By (3.11) the r.h.s. of the equation (3.5) is rewritten as

$$(3.12) \quad \begin{aligned} \frac{\nabla}{\partial\sigma} f(\sigma, u, v, K(\sigma)) = & \left((\alpha + c/2 + \frac{1}{i\hbar}uv) + ((c+1)u + \delta v)\partial_u f + (\delta' u + (c-1)v)\partial_v \right. \\ & \left. + \frac{i\hbar}{4}(\delta(c+1)\partial_u^2 + (\delta\delta' + c^2 - 1)\partial_u \partial_v + \delta'(c-1)\partial_v^2) \right) f(\sigma, u, v, K(\sigma)). \end{aligned}$$

Note also that $a_{2k+1}(\sigma, K)$ satisfies also

$$\frac{\nabla}{d\sigma} a_{2k+1}(\sigma, K(\sigma)) = a_{2k+1}(\sigma, K(\sigma)) *_K (\sigma) : (\alpha + \frac{1}{i\hbar}u \circ v) :_K (\sigma).$$

The equation for this is slightly changed as follows:

$$(3.13) \quad \begin{aligned} \frac{\nabla}{\partial\sigma} f(\sigma, u, v, K(\sigma)) = & \left((\alpha + c/2 + \frac{1}{i\hbar}uv) + ((c-1)u + \delta v)\partial_u f + (\delta' u + (c+1)v)\partial_v \right. \\ & \left. + \frac{i\hbar}{4}(\delta(c-1)\partial_u^2 + (\delta\delta' + c^2 - 1)\partial_u \partial_v + \delta'(c+1)\partial_v^2) \right) f(\sigma, u, v, K(\sigma)). \end{aligned}$$

The difference of solutions of these two equations must satisfies in particular the equation

$$[\frac{1}{i\hbar}u \circ v, f(\sigma, u, v, K(\sigma))]_{*_{K(\sigma)}} = 0.$$

This is a differential equation of order 2 in general:

$$(3.14) \quad [\frac{1}{i\hbar}u \circ v, f(\sigma, u, v, K(\sigma))]_{*_{K(\sigma)}} = \left((u\partial_u - v\partial_v) + \frac{i\hbar}{2}(\delta\partial_u^2 - \delta'\partial_v^2) \right) f(\sigma, u, v, K(\sigma)) = 0.$$

To eliminate the quadratic terms in the second line of (3.12) by the infinitesimal intertwiner, we set

$$\begin{cases} \frac{d}{dt}\delta &= -\delta(c+1) \\ \frac{d}{dt}\delta' &= -\delta'(c-1) \\ \frac{d}{dt}c &= -\frac{1}{2}(\delta\delta' + c^2 - 1) \end{cases}$$

Setting $\xi(t) = \int_0^t c(s)ds$, we have

$$(3.15) \quad \delta(t) = ae^{-\xi(t)-t}, \quad \delta'(t) = a'e^{-\xi(t)+t}, \quad \frac{d^2}{dt^2}\xi(t) = -\frac{1}{2}aa'e^{-2\xi(t)} - \frac{1}{2}\left(\frac{d\xi}{dt}\right)^2 + \frac{1}{2}$$

where a, a' are arbitrary constants. Before solving this, we confirm that these are consistent with (3.10). Plugging the first two equality to (3.10), we have

$$(3.16) \quad \frac{d\xi}{dt} = \pm \sqrt{aa'} e^{-\xi(t)} + \frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}}$$

Differentiating this to obtain

$$\frac{d^2\xi}{dt^2} = -aa' e^{-2\xi(t)} \pm \sqrt{aa'} e^{-\xi(t)} \frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}} + \frac{d}{dt} \left(\frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}} \right).$$

This is the third equation of (3.15). Note that setting $v = \frac{d\xi}{dt}$, the third equation of (3.15) is equivalent with

$$\frac{d}{dt}(v + \sqrt{aa'} e^{-\xi}) = \frac{1}{2} - \frac{1}{2}(v + \sqrt{aa'} e^{-\xi})^2.$$

It follows $v + \sqrt{aa'} e^{-\xi} = \frac{e^{t/2} - e^{-t/2}}{e^{t/2} + e^{-t/2}}$.

We have only to solve (3.16). By setting $\eta(t) = e^{\xi(t)}$, (3.16) gives

$$\frac{d\eta(t)}{dt} = \eta(t) \frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}} \pm \sqrt{aa'}.$$

For simplicity, choose the plus sign $\sqrt{aa'}$. Solving this, we have

$$(3.17) \quad \eta(t) = \left(\gamma - \frac{\sqrt{aa'}}{4}\right)e^t + \left(\gamma + \frac{\sqrt{aa'}}{4}\right)e^{-t} - 2\gamma, \quad \gamma \in \mathbb{C}.$$

Hence using $\eta(t) = e^{\xi(t)}$ we have

$$(3.18) \quad \begin{cases} \delta(t) = ae^{-t}\eta(t)^{-1} = ae^{-\xi(t)-t} \\ \delta'(t) = a'e^t\eta(t)^{-1} = a'e^{-\xi(t)+t} \\ c(t) = \frac{d}{dt} \log \eta(t) = \frac{d}{dt} \xi(t) \end{cases}$$

where a, a' and γ are arbitrarily chosen.

To simplify the solution, we set $\gamma = 0$, $a = a' = 1$ and set in what follows $\eta(t) = -\frac{1}{4}(e^t - e^{-t})$ by restricting the domain t in $\text{Re } t > 0$.

Theorem 3.3 *By choosing a suitable path $K(t)$, equation (3.5) turns out to be a differential equation of order one*

$$\partial_t f(t, K(t); u, v) = \left(\left(\frac{c}{2} + \alpha + \frac{1}{i\hbar} uv \right) + ((c+1)u + \delta v) \partial_u + (\delta' u + (c-1)v) \partial_v \right) f(t, K(t); u, v).$$

Changing variables $x = \frac{1}{\sqrt{i\hbar}}u$, $y = \frac{1}{\sqrt{i\hbar}}v$ and setting

$$C(t) = \left(\alpha + \frac{c(t)}{2}\right), \quad L(t) = \begin{bmatrix} c(t)+1 & \delta'(t) \\ \delta(t) & c(t)-1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (K(t) + J),$$

gives the equivalent equation

$$(3.19) \quad \left(\partial_t - (x, y)L(t) \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} \right) f(t, K(t); x, y) = (xy + C(t))f(t, K(t); x, y).$$

As we set $\eta(t) = -\frac{1}{4}(e^t - e^{-t})$,

$$C(t) = \alpha + \frac{1}{2} \frac{e^t + e^{-t}}{e^t - e^{-t}}, \quad L(t) = \begin{bmatrix} \frac{2}{1-e^{-2t}} & 4 \frac{1}{1-e^{2t}} \\ 4 \frac{-1}{1-e^{-2t}} & \frac{2}{1-e^{2t}} \end{bmatrix}$$

To solve (3.19), we set $\psi_\lambda(t) = \int_\lambda^t \frac{1}{e^{2s}-1} ds$ for every positive λ . Easily, $\int_\lambda^t \frac{1}{1-e^{-2s}} ds = t - \lambda + \psi_\lambda(t)$, and if t is real then

$$(t - \lambda + \psi_\lambda(t))\psi_\lambda(t) \geq 0, \quad \text{and} \quad > 0, (t \neq \lambda).$$

Choosing different positive real numbers λ and μ , we set

$$\tilde{L}(t) = \begin{bmatrix} t - \mu + \psi_\mu(t) & \psi_\lambda(t) \\ -t + \lambda - \psi_\lambda(t) & \psi_\mu(t) \end{bmatrix}.$$

Then, we see $\frac{d}{dt} \tilde{L}(t) = L(t)$ and $\det \tilde{L}(t)$ is positive definite on $t > 0$.

Letting $g(t, x, y) = f(t, (x, y)\tilde{L}(t))$, we see

$$\frac{d}{dt} g(t, x, y) = (xy + C(t))g(t, x, y), \quad \text{hence} \quad g(t, x, y) = e^{txy + \tilde{C}(t)} G(x, y)$$

where $\tilde{C}(t) = \log(e^{\alpha t}(e^t - e^{-t}))$ and $G(x, y)$ is an arbitrary holomorphic function. Viewing $\phi_t = \tilde{L}^{-1}(t) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ as a linear diffeomorphism, $f(t, x, y)$ is obtained by the pull-back of $g(t, x, y)$:

$$f(t, x, y) = \phi_t^*(e^{txy + \tilde{C}(t)} G(x, y)).$$

Since $:e_*^{(t+s^2)(\alpha + \frac{1}{i\hbar} u \circ v)} :_{K(t)}$ satisfies this equation,

$$:e_*^{(t+s^2)(\alpha + \frac{1}{i\hbar} u \circ v)} :_{K(t)} = \phi_t^*(e^{t \frac{1}{i\hbar} uv + \tilde{C}(t)} G(u, v)).$$

For adjusting the initial data, setting $t = -s^2$, we see

$$1 = \phi_{-s^2}^*(e^{-s^2 \frac{1}{i\hbar} uv + \tilde{C}(-s^2)} G(u, v))$$

$$1 = \tilde{L}(-s^2)(1) = e^{-s^2 \frac{1}{i\hbar} uv + \tilde{C}(-s^2)} G(u, v)$$

It follows that $G(u, v) = e^{s^2 \frac{1}{i\hbar} uv - \tilde{C}(-s^2)}$. Plugging this we have

$$:e_*^{(t+s^2)(\alpha + \frac{1}{i\hbar} u \circ v)} :_{K(t)} = \phi_t^*(e^{(t+s^2) \frac{1}{i\hbar} uv + \tilde{C}(t) - \tilde{C}(-s^2)}) = \phi_t^*(e^{(t+s^2)(\frac{1}{i\hbar} uv + \alpha) + \frac{1}{2}\xi(t) - \frac{1}{2}\xi(-s^2)}).$$

This may be written as

$$:e_*^{(t+s^2)(\alpha + \frac{1}{i\hbar} u \circ v)} :_{K(t)} = \sqrt{\frac{\eta(t)}{\eta(-s^2)}} \phi_t^*(e^{(t+s^2)(\frac{1}{i\hbar} uv + \alpha)}) = \sqrt{\frac{e^{-t} - e^t}{e^{s^2} - e^{-s^2}}} \phi_t^*(e^{(t+s^2)(\frac{1}{i\hbar} uv + \alpha)}).$$

If we give an attention to a movement of specific singular point and use a suitable expression parameter moving together with the singular point, then the *-exponential function looks as if it were a classical function $e^{(t+s^2)(\frac{1}{i\hbar} uv + \alpha)}$.

4 Matrix elements in generic ordered expressions

So far, we are concerned with individual/ subjective singular point. This depends on the individual expression parameters. However, there are a lot of mathematical facts which can be stated without specifying the expression parameters. Such one may be viewed as “objective” object that is commonly accepted by many observers. That is, we are thinking that the “objective” object is nothing but objects which almost all expression parameters accept. In this section, we discuss such common objects.

First of all, we note several identities that ensure the associativity:

Lemma 4.1 *If $p \neq 0$, then $\varpi_{00}*(u^p*\varpi_{00})=0$, and $(\varpi_{00}*v^p)*\varpi_{00}=0$.*

Proof By taking the formal power series expansion with respect to $i\hbar$ for $e_*^{su* v}$, the formal associativity theorem (cf.[11]) together with the bumping identity gives the following:

$$e_*^{su* v}*(u^p*e_*^{tu* v})=(e_*^{su* v}*u^p)*e_*^{tu* v}=u^p*e_*^{(s+t)u* v+i\hbar ps}.$$

The right hand side of the above equality is continuous in s, t . In particular,

$$\lim_{t \rightarrow a} e_*^{su* v}*(u^p*e_*^{tu* v})=e_*^{su* v}*\lim_{t \rightarrow a} (u^p*e_*^{tu* v}).$$

Using the bumping identity, we have

$$\begin{aligned} e_*^{su* v}*(u^p*\lim_{t \rightarrow -\infty} e_*^{tu* v}) &= e_*^{su* v}*\lim_{t \rightarrow -\infty} u^p*e_*^{tu* v} = \lim_{t \rightarrow -\infty} u^p*e_*^{(s+t)u* v+i\hbar ps} \\ &= u^p*\lim_{t \rightarrow -\infty} e_*^{(s+t)u* v+i\hbar ps} = u^p e^{i\hbar ps}*\varpi_{00}. \end{aligned}$$

It follows that

$$\varpi_{00}*(u^p*\varpi_{00}) = \lim_{s \rightarrow -\infty} e_*^{s\frac{1}{i\hbar}u* v}*(\lim_{t \rightarrow -\infty} u^p*e_*^{t\frac{1}{i\hbar}u* v}) = \lim_{s \rightarrow -\infty} u^p e^{ps}*\varpi_{00} = 0.$$

Similarly, we also have $(\varpi_{00}*v^p)*\varpi_{00}=0$.

These are proved also by using the integral expressions.

$$\begin{aligned} \int_0^{2\pi} e_*^{(s+it)\frac{1}{i\hbar}u* v} dt * u^p * \int_0^{2\pi} e_*^{(s+i\tau)\frac{1}{i\hbar}u* v} d\tau &= u^p * \int_0^{2\pi} e_*^{(s+it)(\frac{1}{i\hbar}u* v+p)} dt * \int_0^{2\pi} e_*^{(s+i\tau)\frac{1}{i\hbar}u* v} d\tau \\ &= u^p * \int_0^{2\pi} e^{(s+it)p} e_*^{(s+it)\frac{1}{i\hbar}u* v} dt * \int_0^{2\pi} e_*^{(s+i\tau)\frac{1}{i\hbar}u* v} d\tau = u^p * \int_0^{2\pi} \int_0^{2\pi} e^{(s+it)p} e_*^{(2s+it+i\tau)\frac{1}{i\hbar}u* v} dt d\tau. \end{aligned}$$

Thus, the changing variables gives that this vanishes by $\int_0^{2\pi} e^{(s+it)p} dt = 0$. □

Lemma 4.2 *For every polynomial $f(u, v) = \sum a_{ij} u^i * v^j$,*

$$\varpi_{00}*(f(u, v)*\varpi_{00}) = f(0, 0)\varpi_{00} = (\varpi_{00}*f(u, v))*\varpi_{00}.$$

*Consequently, associativity holds for $\varpi_{00}*f(u, v)*\varpi_{00}$ for a polynomial $f(u, v)$.*

By the formal associativity theorem (cf.[11]), we have easily

$$\begin{aligned} (e_*^{su*} * v^q) * (u^p * e_*^{tu*}) &= e_*^{su*} * (v^q * u^p * e_*^{tu*}) = e^{(q-p)t\hbar} e_*^{(s+t)u*} * v^q * u^p \quad \text{for } q \geq p, t \\ (e_*^{su*} * v^q) * (u^p * e_*^{tu*}) &= e_*^{su*} * (v^q * u^p * e_*^{tu*}) = v^q * u^p * e_*^{(s+t)u*} * e^{(p-q)s\hbar} \quad \text{for } q \leq p. \end{aligned}$$

Replacing s, t by $\frac{1}{i\hbar}s, \frac{1}{i\hbar}t$ and taking $\lim_{t \rightarrow -\infty}, \lim_{s \rightarrow -\infty}$ for the case $p \geq q, q \geq p$ respectively, we have

$$(4.1) \quad (\varpi_{00} * v^q) * (u^p * \varpi_{00}) = \delta_{p,q} p! (i\hbar)^p = \varpi_{00} * (v^q * u^p * \varpi_{00}) = (\varpi_{00} * v^q * u^p) * \varpi_{00}.$$

Since $\varpi_{00} * v^q * u^p * \varpi_{00} = \delta_{p,q} p! (i\hbar)^p \varpi_{00}$, we have the following:

Proposition 4.1 *In generic ordered expressions, $E_{p,q} = \frac{1}{\sqrt{p!q!(i\hbar)^{p+q}}} u^p * \varpi_{00} * v^q$ is the (p, q) -matrix element, that is $E_{p,q} * E_{r,s} = \delta_{q,r} E_{p,s}$. The K -expression $:E_{p,q}:_K$ of $E_{p,q}$ will be denoted by $E_{p,q}(K)$. Note that $E_{0,0}(K) = : \varpi_{00} :_K$.*

Similar calculation caring the \pm sign shows also

Proposition 4.2 *$\bar{E}_{p,q} = \frac{\sqrt{-1}^{p+q}}{\sqrt{p!q!(i\hbar)^{p+q}}} v^p * \bar{\varpi}_{00} * u^q$ is the (p, q) -matrix element in generic ordered expressions. The K -expression of $\bar{E}_{p,q}$ will be denoted by $\bar{E}_{p,q}(K)$. Note that $\bar{E}_{0,0}(K) = : \bar{\varpi}_{00} :_K$.*

Proposition 2.2 gives

$$(4.2) \quad E_{p,q} * \bar{E}_{r,s} = 0 = \bar{E}_{r,s} * E_{p,q}.$$

By a similar computation, we can consider the idempotent element $\varpi_*(0) = \frac{1}{2\pi} \int_0^{2\pi} e_*^{it(\frac{1}{i\hbar}u \circ v)} dt$ in Theorem 2.1, called pseudo-vacuum. We first recall some formulas which will be used below. The bumping identity gives

$$v * (v * u) * u = v * (u \circ v + \frac{1}{2}i\hbar) * u = (u \circ v + \frac{1}{2}i\hbar) * (u \circ v + \frac{3}{2}i\hbar).$$

Repeating this we see that

$$\begin{aligned} (4.3) \quad \frac{1}{(i\hbar)^n} v^n * u^n &= \left(\frac{1}{i\hbar} u \circ v + \frac{1}{2}\right) * \left(\frac{1}{i\hbar} u \circ v + \frac{3}{2}\right) * \cdots * \left(\frac{1}{i\hbar} u \circ v + \frac{2n-1}{2}\right) \\ &= \left(\frac{1}{i\hbar} v * u\right) * \left(\frac{1}{i\hbar} v * u + 1\right) * \cdots * \left(\frac{1}{i\hbar} v * u + n - 1\right) = \left\{ \frac{1}{i\hbar} v * u \right\}_{*n}, \end{aligned}$$

where $\{A\}_{*n} = A * (A+1) * \cdots * (A+n-1)$, $\{A\}_{*0} = 1$. Next formulas are very useful in our computations

$$\left\{ \frac{1}{i\hbar} u * v \right\}_{*n} * u = u * \left\{ \frac{1}{i\hbar} u * v + 1 \right\}_{*n}, \quad \left\{ \frac{1}{i\hbar} u * v \right\}_{*n} * v = v * \left\{ \frac{1}{i\hbar} u * v - 1 \right\}_{*n}.$$

Note that the identity $(u \circ v) * \varpi_*(0) = 0$ gives

$$(4.4) \quad \left(\frac{1}{i\hbar} u * v + \ell\right) * \varpi_*(0) = \left(\frac{1}{i\hbar} u \circ v + \ell - \frac{1}{2}\right) * \varpi_*(0) = \left(\ell - \frac{1}{2}\right) \varpi_*(0).$$

It follows that

$$\begin{aligned}\varpi_*(0) * \left(\frac{1}{i\hbar}\right)^n u^n * v^n * \varpi_*(0) &= (1/2)_n \varpi_*(0) \\ \varpi_*(0) * \left(\frac{1}{i\hbar}\right)^n v^n * u^n * \varpi_*(0) &= (1/2)_{-n} \varpi_*(0),\end{aligned}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 = 1$ and $(a)_{-n} = (a-1)(a-2)\cdots(a-n)$. If we use the convention (2.18), then this is written by

$$(4.5) \quad \varpi_*(0) * \left(\frac{1}{i\hbar}\right)^n \hat{\zeta}^n * \zeta^n * \varpi_*(0) = \left(\frac{1}{2}\right)_n \varpi_*(0), \quad n \in \mathbb{Z},$$

where ζ^n , $\hat{\zeta}^n$ are given by (2.18).

Lemma 4.3 *If $K \in \mathfrak{K}_0$, then $:e_*^{it(\frac{1}{i\hbar}u \circ v)}:_K$ is 2π -periodic and*

$$D_{k,\ell}(K) = \frac{1}{\sqrt{(\frac{1}{2})_k (\frac{1}{2})_\ell (i\hbar)^{|k|+|\ell|}}} \zeta^k * : \varpi_*(0) :_K * \hat{\zeta}^\ell, \quad : \varpi_*(0) :_K = \frac{1}{2\pi} \int_0^{2\pi} : e_*^{it(\frac{1}{i\hbar}u \circ v)} dt :_K dt$$

are matrix elements for every $k, \ell \in \mathbb{Z}$. Note that $D_{n,n}(K) = \frac{1}{2\pi} \int_0^{2\pi} : e_*^{it(\frac{1}{i\hbar}u \circ v - n)} :_K dt$.

Proof Different from the ordinary vacuum or bar-vacuum, we see $u * \varpi_*(0) \neq 0$, $v * \varpi_*(0) \neq 0$. But note that the bumping identity gives

$$(4.6) \quad u^n * e_*^{it(\frac{1}{i\hbar}u \circ v)} = e_*^{it(\frac{1}{i\hbar}u \circ v - n)} * u^n, \quad v^n * e_*^{it(\frac{1}{i\hbar}u \circ v)} = e_*^{it(\frac{1}{i\hbar}u \circ v + n)} * v^n.$$

Moreover, if $k \neq \ell$, then the exponential law and the change of variables gives

$$\int_0^{2\pi} e_*^{is(\frac{1}{i\hbar}u \circ v + k)} ds * \int_0^{2\pi} e_*^{it(\frac{1}{i\hbar}u \circ v + \ell)} dt = \int_0^{2\pi} e^{it(k-\ell)} dt \int_0^{2\pi} e_*^{is(\frac{1}{i\hbar}u \circ v + \ell)} ds = 0,$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} e_*^{it(\frac{1}{i\hbar}u \circ v + k)} dt * \frac{1}{2\pi} \int_0^{2\pi} e_*^{is(\frac{1}{i\hbar}u \circ v + k)} ds = \frac{1}{2\pi} \int_0^{2\pi} e_*^{it(\frac{1}{i\hbar}u \circ v + k)} dt.$$

Theorem 2.1, (2.16) and (2.19) show that the $*$ -product $P(u, v) * \varpi_*(0) * Q(u, v)$ by any polynomials $P(u, v)$, $Q(u, v)$ is reduced to the shape $\phi * \varpi_*(0) * \psi$ where ϕ, ψ are polynomials of single variable u or v .

Using the above formula, (4.5), (4.6), we have the desired result. \square

Since $\frac{1}{2\pi} \int_0^{2\pi} : e_*^{is(\frac{1}{i\hbar}u \circ v - n)} :_K ds = D_{n,n}(K)$, the Fourier expansion of $e_*^{it(\frac{1}{i\hbar}u \circ v)}$ is written as

$$(4.7) \quad : e_*^{it(\frac{1}{i\hbar}u \circ v)} :_K = \frac{1}{2\pi} \sum_n \int_0^{2\pi} : e_*^{is(\frac{1}{i\hbar}u \circ v - n)} :_K ds e^{int} = \sum_{n \in \mathbb{Z}} D_{n,n}(K) e^{int}, \quad K \in \mathfrak{K}_0.$$

Hence we see by the exponential law for every α , the Fourier series

$$(4.8) \quad : e_*^{it(\frac{1}{i\hbar}u \circ v + \alpha)} :_K = \sum_{n \in \mathbb{Z}} D_{n,n}(K) e^{it(n+\alpha)}, \quad K \in \mathfrak{K}_0$$

converges uniformly in $C^\infty(S^1, Hol(\mathbb{C}^2))$, the C^∞ -topology of the space of $Hol(\mathbb{C}^2)$ -valued smooth functions on S^1 .

4.1 Matrix representations

Now back to the case ϖ_{00} , we note that one can set $w=e^t$ in the l.h.s. of (1.7). As $u*v = u \circ v - \frac{1}{2}i\hbar$, Proposition 2.1 gives the convergence

$$\lim_{w \rightarrow 0} :e_*^{\log w \frac{1}{i\hbar} 2u*v} :_K = :e_*^{\log 0 \frac{1}{i\hbar} 2u*v} :_K = : \varpi_{00} :_K.$$

Hence, we get a holomorphic function of w defined on an open neighborhood of $w = 0$.

$$f_K(w) = :e_*^{\log w \frac{1}{i\hbar} u*v} :_K.$$

Using the bumping identity, we have

$$\partial_w \Big|_0 e_*^{(\log w) \frac{1}{i\hbar} u*v} = \lim_{t \rightarrow -\infty} e^{-t} \partial_t e_*^{t \frac{1}{i\hbar} u*v} = \lim_{t \rightarrow -\infty} \frac{1}{i\hbar} u*v e_*^{t(\frac{1}{i\hbar} u*v - 1)} = \lim_{t \rightarrow -\infty} \frac{1}{i\hbar} u * e_*^{t(\frac{1}{i\hbar} u*v)} * v.$$

Repeating this procedure, we have the following remarkable formulas:

$$(4.9) \quad \frac{1}{n!} f_K^{(n)}(0) = \frac{1}{n! (i\hbar)^n} :u^n * \varpi_{00} * v^n :_K.$$

The convergence of Taylor series gives $f_K(w) = \sum_{n=0}^{\infty} \frac{1}{n!} f_K^{(n)}(0) w^n$. These are in (n, n) -matrix elements, denoted by $:E_{n,n} :_K$. It follows in generic ordered expressions,

$$e_*^{\log w \frac{1}{i\hbar} u*v} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{i\hbar} u*v \right)^n * e^{-\frac{1}{i\hbar} u*v} w^n = \sum_{n=0}^{\infty} \frac{1}{n! (i\hbar)^n} u^n * \varpi_{00} * v^n w^n = \sum_{n \geq 0} w^n E_{n,n}$$

on a neighborhood of $w = 0$ depending on the expression parameter K . In what follows we use notations

$$(4.10) \quad \frac{1}{n! (i\hbar)^n} u^n * \varpi_{00} * v^n :_K = :E_{n,n} :_K = E_{n,n}(K).$$

In particular,

$$:E_{0,0} :_K = E_{0,0}(K) = : \varpi_{00} :_K$$

Here we apply the following general fact:

Lemma 4.4 *If $f(z)$ is a holomorphic mapping from a complex open disk $D(R)$ of radius R centered at 0 into a Fréchet space over \mathbb{C} , then the Taylor series of $f(z)$ at $z=0$ converges uniformly on any closed disk of radius r , $0 < r < R$.*

Proposition 4.3 *If the singular points of $:e_*^{t \frac{1}{i\hbar} u*v} :_K$ lies on the open right half plane, then we have in the space $Hol(\mathbb{C}^2)$*

$$1 = \sum_{n=0}^{\infty} :E_{n,n} :_K, \quad :e_*^{\tau \frac{1}{i\hbar} u*v} :_K = \sum_{n=0}^{\infty} e^{n\tau} :E_{n,n} :_K, \quad (\operatorname{Re} \tau \leq 0).$$

Since $\frac{1}{i\hbar} u*v = u \circ v - \frac{1}{2}$, the second equality may be written as

$$:e_*^{\tau \frac{1}{i\hbar} u \circ v} :_K = \sum_{n=0}^{\infty} e^{\tau(n+\frac{1}{2})} E_{n,n}(K), \quad (\operatorname{Re} \tau \leq 0).$$

Similarly, set $w = e^t$ in the l.h.s. of (1.7) by noting $v*u = u \circ v + \frac{1}{2}i\hbar$. Proposition 2.1 gives the convergence

$$\lim_{w \rightarrow \infty} :e_*^{\log w \frac{1}{i\hbar} 2v*u} :_K$$

which we called a **bar-vacuum** and denoted by $\overline{\varpi}_{00}$. Hence we get a holomorphic function of $\hat{w} = w^{-1}$ defined on a neighborhood of $\hat{w} = 0$ depending on K . Set as follows:

$$g_K(\hat{w}) = :e_*^{-\log \hat{w} \frac{1}{i\hbar} v*u} :_K$$

Using the bumping identity, we have

$$\partial_{\hat{w}} \Big|_0 e_*^{-(\log \hat{w}) \frac{1}{i\hbar} v*u} = - \lim_{t \rightarrow \infty} e^t \partial_t e_*^{t \frac{1}{i\hbar} v*u} = - \lim_{t \rightarrow \infty} \frac{1}{i\hbar} v*u e_*^{t(\frac{1}{i\hbar} v*u + 1)} = - \lim_{t \rightarrow \infty} \frac{1}{i\hbar} v e_*^{t(\frac{1}{i\hbar} v*u)} * u.$$

Repeating this procedure, we have the following remarkable formulas:

$$(4.11) \quad \frac{1}{n!} g_K^{(n)}(0) = \frac{(-1)^n}{n!(i\hbar)^n} :v^n * \overline{\varpi}_{00} * u^n :_K = : \overline{E}_{n,n} :_K (= \overline{E}_{n,n}(K)).$$

It follows in generic ordered expression

$$e_*^{-\log \hat{w} \frac{1}{i\hbar} v*u} = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!(i\hbar)^n} :v^n * \overline{\varpi}_{00} * u^n :_K \right) \hat{w}^n, \quad \hat{w} = w^{-1}.$$

Theorem 4.1 *Let $I_{\circ}(K) = [a, b]$ be the exchanging interval. Then the radius of convergence of the Taylor series in the Fréchet space $\text{Hol}(\mathbb{C}^2)$ of*

$$f_K(w) = \sum \frac{1}{n!} f_K^{(n)}(0) w^n, \quad (\text{resp. } g_K(\hat{w}) = \sum \frac{1}{n!} g_K^{(n)}(0) \hat{w}^n, \quad \hat{w} = w^{-1})$$

is e^a (resp. e^{-b}). If $K \in \mathfrak{K}_+$ (resp. \mathfrak{K}_-), then the radius of convergence is bigger than 1.

Proposition 4.4 *In the K -ordered expression for $K \in \mathfrak{K}_+$, we have*

$$1 = \sum_{n=0}^{\infty} E_{n,n}(K), \quad :e_*^{\frac{\tau}{i\hbar}(u \circ v + \lambda)} :_K = \sum_{n=0}^{\infty} e^{\tau(n + \frac{1}{2}) + \tau\lambda} E_{n,n}(K), \quad (\text{Re } \tau \leq 0).$$

The former converges in the space $\text{Hol}(\mathbb{C}^2)$. If $\text{Re } \tau \leq 0$, the latter converges uniformly on every compact subset w.r.t. τ . Similarly, if $K \in \mathfrak{K}_-$, then

$$1 = \sum_{n=0}^{\infty} \overline{E}_{n,n}(K), \quad :e_*^{\frac{\tau}{i\hbar}(u \circ v + \lambda)} :_K = \sum_{n=0}^{\infty} e^{\tau(-n - \frac{1}{2}) + \tau\lambda} \overline{E}_{n,n}(K), \quad (\text{Re } \tau \geq 0).$$

The former converges in the space $\text{Hol}(\mathbb{C}^2)$. If $\text{Re } \tau \geq 0$, the latter converges uniformly on every compact subset w.r.t. τ .

Note that the conditions $K \in \mathfrak{K}_+$ and $\operatorname{Re} \tau \leq 0$ are used only to ensure that $\sum_{n=0}^{\infty} E_{n,n}(K)$ and $\sum e^{\tau(n+\frac{1}{2})+\tau\lambda} E_{n,n}(K)$ in the r.h.s. converge in the topology of $\operatorname{Hol}(\mathbb{C}^2)$. However, each term of $\sum_{n=0}^{\infty} e^{\tau(n+\frac{1}{2})+\tau\lambda} E_{n,n}(K)$ is an entire function of τ , although $:e_*^{\tau\frac{1}{i\hbar}u\circ v+\tau\lambda}:_K$ may be singular at some $\tau=\tau_0$. Singular point $\tau=\tau_0$ means simply that $\sum_{n=0}^{\infty} e^{\tau_0(n+\frac{1}{2})+\tau_0\lambda} E_{n,n}(K)$ diverges in the topology of $\operatorname{Hol}(\mathbb{C}^2)$.

Note that without conditions such as $K \in \mathfrak{K}_+$ and $\operatorname{Re} t \leq 0$, $:e_*^{\log w \frac{1}{i\hbar}u*v}:_K$ is holomorphic on a neighborhood of $w = 0$. Hence setting $e_*^{c\frac{1}{i\hbar}u*v} * e_*^{(z-c)\frac{1}{i\hbar}u*v}$ by choosing c sufficiently large, the Taylor series $e_*^{(z-c)\frac{1}{i\hbar}u*v} = \sum_{n \geq 0} e^{(z-c)(n+\frac{1}{2})} E_{n,n}$ of $e_*^{(z-c)\frac{1}{i\hbar}u*v}$ at $w = 0$ converges in $\operatorname{Hol}(\mathbb{C}^2)$. Hence, applying $e_*^{c\frac{1}{i\hbar}u*v}$, we have

$$e_*^{z\frac{1}{i\hbar}u*v} = e_*^{c\frac{1}{i\hbar}u*v} * \sum_{n \geq 0} e^{(z-c)(n+\frac{1}{2})} E_{n,n}.$$

Although $e_*^{c\frac{1}{i\hbar}u*v} *$ of r.h.s. is not continuous in $\operatorname{Hol}(\mathbb{C}^2)$, the componentwise computation of matrices may be applied to yield

$$e_*^{z\frac{1}{i\hbar}u*v} = e_*^{c\frac{1}{i\hbar}u*v} * \sum_{n \geq 0} e^{(z-c)(n+\frac{1}{2})} E_{n,n} = \sum_{n \geq 0} e^{z(n+\frac{1}{2})} E_{n,n}$$

as $\frac{1}{i\hbar}u*v * E_{n,n} = nE_{n,n}$ gives $e_*^{c\frac{1}{i\hbar}u*v} * E_{n,n} = e^{cn} E_{n,n}$. We denote this by

$$(4.12) \quad :e_*^{z\frac{1}{i\hbar}u*v}:_{E(K)_{mat}} = \sum_{n \geq 0} e^{z(n+\frac{1}{2})} E_{n,n}(K).$$

Similarly, we have

$$e_*^{z\frac{1}{i\hbar}v*u} = e_*^{-c\frac{1}{i\hbar}v*u} * \sum_{n \geq 0} e^{-(z+c)(n+\frac{1}{2})} \overline{E}_{n,n} = \sum_{n \geq 0} e^{-z(n+\frac{1}{2})} \overline{E}_{n,n}$$

and this is denoted by

$$(4.13) \quad :e_*^{z\frac{1}{i\hbar}v*u}:_{\overline{E}(K)_{mat}} = \sum_{n \geq 0} e^{-z(n+\frac{1}{2})} \overline{E}_{n,n}(K).$$

4.1.1 Representations by Laurent expansions

If $a < \operatorname{Re} z < b$, then setting $w = e^z$ in (1.7), Laurent expansion of $:e_*^{\log w \frac{1}{i\hbar}u\circ v}:_K$ by w gives that

$$(4.14) \quad :e_*^{z(\frac{1}{i\hbar}u\circ v)}:_K = \sum_{\ell=-\infty}^{\infty} \tilde{D}_{\ell,\ell}(K) e^{z\ell}, \quad a < \operatorname{Re} z < b$$

and the r.h.s. converges in $\operatorname{Hol}(\mathbb{C}^2)$. Note that

$$\tilde{D}_{n,n}(K) = \frac{1}{2\pi} \int_0^{2\pi} :e_*^{(s+it)(\frac{1}{i\hbar}u\circ v)}:_K e^{-(s+it)n} dt, \quad a < s < b.$$

Cauchy's integral theorem shows that $\tilde{D}_{n,n}(K)$ is independent of s whenever $s \in I_0(K)$.

In particular, if $a < 0 < b$, we see that $\tilde{D}_{n,n}(K) = D_{n,n}(K)$ the (n, n) diagonal matrix element given in Lemma 4.3. Hence

Proposition 4.5 *If $K \in \mathfrak{K}_0$, then $\tilde{D}_{n,n}(K)$ is an (n, n) diagonal matrix element and for τ such that $a < \operatorname{Re} \tau < b$,*

$$:e_*^{\tau \frac{1}{i\hbar}(u \circ v + \lambda)}:_K = \sum_{n=-\infty}^{\infty} e^{\tau n + \tau \lambda} \tilde{D}_{n,n}(K), \quad \lambda \in \mathbb{C},$$

and the r.h.s. converges in $\operatorname{Hol}(\mathbb{C}^2)$. In particular, $1 = \sum_{n=-\infty}^{\infty} \tilde{D}_{n,n}(K)$.

Although $\tilde{D}_{n,n}(K) *_K \tilde{D}_{n,n}(K)$ is not defined in general, we show in what follows that $\tilde{D}_{n,n}(K)$ may be viewed as a diagonal matrix element.

Note now

$$\frac{1}{i\hbar}(u \circ v) * \int_0^{2\pi} e_*^{(s+i\tau)\frac{1}{i\hbar}u \circ v} e^{-(s+i\tau)n} d\tau = n \int_0^{2\pi} e_*^{(s+i\tau)\frac{1}{i\hbar}u \circ v} e^{-(s+i\tau)n} d\tau.$$

It follows that $:e_*^{-s\frac{1}{i\hbar}u \circ v}:_K *_K \tilde{D}_{n,n}(K) = e^{-ns} \tilde{D}_{n,n}(K)$. Hence one may write

$$:e_*^{it\frac{1}{i\hbar}u \circ v}:_K = \sum_{n=-\infty}^{\infty} \tilde{D}_{n,n}(K) e^{int}$$

though r.h.s. does not converge in $\operatorname{Hol}(\mathbb{C}^2)$. To avoid the confusion, we denote this by

$$(4.15) \quad :e_*^{it\frac{1}{i\hbar}u \circ v}:_{{D(K)}_{mat}} = \sum_{n=-\infty}^{\infty} \tilde{D}_{n,n}(K) e^{int}$$

and regard this as a formal element. We see that $\tilde{D}_{n,n}(K) = \frac{1}{2\pi} \int_0^{2\pi} :e_*^{it\frac{1}{i\hbar}u \circ v}:_{{D(K)}_{mat}} e^{-itn} dt$. Suppose the exponential law holds formally on the l.h.s., then one must set

$$\tilde{D}_{m,m}(K) *_{{mat}} \tilde{D}_{n,n}(K) = \delta_{m,n} \tilde{D}_{n,n}(K)$$

where $*_{mat}$ indicates the formal product which allows the formal exponential law. By this observation, we define $:e_*^{it\frac{1}{i\hbar}u \circ v}:_{{D(K)}_{mat}}$ as a diagonal matrix, and $\tilde{D}_{n,n}(K)$ as a diagonal matrix element.

By (4.12), (4.13) and (4.15), $e_*^{it\frac{1}{i\hbar}u \circ v}$ is expressed as diagonal matrices, which will be called **diagonal matrix expressions**.

4.1.2 Fourier series of alternating 2π -periodic functions

Beside these, we note that the matrix representation via Taylor expansion in the previous section can be obtained by using the Fourier expansion along a closed curve parallel to the pure imaginary axis (cf.(2.3)).

Every 4π -periodic function $f(\theta)$ is written as the sum of a 2π -periodic and a alternating 2π -periodic functions:

$$f(\theta) = f_0(\theta) + f_-(\theta), \quad f_0(\theta) = \frac{1}{2}(f(\theta) + f(\theta + 2\pi)), \quad f_-(\theta) = \frac{1}{2}(f(\theta) - f(\theta + 2\pi)).$$

The Fourier series of $f(\theta)$ is given as

$$\begin{aligned} f(\theta) &= \sum_n \frac{1}{4\pi} \int_0^{4\pi} f(t) e^{-\frac{1}{2}int} dt e^{i\frac{n}{2}\theta} \\ &= \frac{1}{4\pi} \sum_n \frac{1}{4\pi} \int_0^{4\pi} f_0(t) e^{-\frac{1}{2}i2nt} dt e^{in\theta} + \sum_n \frac{1}{4\pi} \int_0^{4\pi} f_-(t) e^{-\frac{1}{2}i(2n+1)t} dt e^{i(n+\frac{1}{2})\theta}. \end{aligned}$$

As the periodicity of $:e_*^{(s+it)\frac{1}{i\hbar}u\circ v}$ w.r.t. t depends on s , we have to use Fourier basis depending on s . Hence we have

Theorem 4.2 $e_*^{(s+it)(\frac{1}{i\hbar}u\circ v+z)}$ is expressed by diagonal matrices

$$\begin{aligned} :e_*^{(s+it)(\frac{1}{i\hbar}u\circ v+z)}:_{E(K)mat} &= \sum_{k=0}^{\infty} E_{k,k}(K) e^{(s+it)(k+\frac{1}{2}+z)}, \quad z \in \mathbb{C} \\ :e_*^{(s+it)(\frac{1}{i\hbar}u\circ v+z)}:_{D(K)mat} &= \sum_{n=-\infty}^{\infty} \tilde{D}_{n,n}(K) e^{(s+it)(n+z)}, \quad z \in \mathbb{C} \\ :e_*^{(s+it)(\frac{1}{i\hbar}u\circ v+z)}:_{\bar{E}(K)mat} &= \sum_{k=0}^{\infty} \bar{E}_{k,k}(K) e^{(s+it)(-k-\frac{1}{2}+z)}, \quad z \in \mathbb{C} \end{aligned}$$

where $E_{k,k}(K)$, $\bar{E}_{k,k}(K)$ equal to the ones given by (4.10), (4.11). Let $I_o(K) = [a, b]$ be the exchanging interval in generic ordered expression K . Then, these converge for $s < a$, $a < s < b$ and $b < s$ respectively in $C^\infty(S^1, Hol(\mathbb{C}^2))$ ($Hol(\mathbb{C}^2)$ -valued smooth functions on S^1 with C^∞ -topology).

For every Schwartz distribution $f(t)$ on S^1 , the integral

$$:\tilde{f}_s(z + \frac{1}{i\hbar}u\circ v):_K = \int_{S^1} f(t) :e_*^{(s+it)(z + \frac{1}{i\hbar}u\circ v)}:_K dt, \quad \text{for generic } K$$

gives an element of $Hol(\mathbb{C}^2)$. $\tilde{f}_s(z + \frac{1}{i\hbar}u\circ v)$ has a diagonal matrix expression as follow:

$$\begin{aligned} :\tilde{f}_s(z + \frac{1}{i\hbar}u\circ v):_{E(K)mat} &= \sum_{n=0}^{\infty} \int_{S^1} f(t) e^{(s+it)(z + \frac{1}{2} + n)} dt E_{n,n}(K), \\ :\tilde{f}_s(z + \frac{1}{i\hbar}u\circ v):_{D(K)mat} &= \sum_{n=-\infty}^{\infty} \int_{S^1} f(t) e^{(s+it)(z + n)} dt \tilde{D}_{n,n}(K), \\ :\tilde{f}_s(z + \frac{1}{i\hbar}u\circ v):_{\bar{E}(K)mat} &= \sum_{n=0}^{\infty} \int_{S^1} f(t) e^{(s+it)(z - \frac{1}{2} - n)} dt \bar{E}_{n,n}(K) \end{aligned}$$

The series of r.h.s. converge respectively $s < a$, $a < s < b$, $b < s$ in the space $Hol(\mathbb{C}^2)$, where $(a, b) = I_o(K)$ is the exchanging interval of $:e_*^{\tau\frac{1}{i\hbar}u\circ v}:_K$.

More systematic treatment of the diagonal matrix expressions will be given in the next paper

4.2 Diagonal matrix expressions and applications

Note that in Theorem 4.2 one may set $s = 0$, when $0 < a$, $a < 0 < b$ and $b < 0$ respectively. Thus, by differentiating k -times and by setting $s+it=0$ in each case, an element such as $(z + \frac{1}{i\hbar}u\circ v)_*^k$, $k \geq 0$, has three expressions as diagonal matrices in the space $Hol(\mathbb{C}^2)$:

$$(4.16) \quad (z + \frac{1}{i\hbar}u\circ v)_*^k :_K = \begin{cases} \sum_{n=0}^{\infty} (z + n + \frac{1}{2})^k E_{n,n}(K), & K \in \mathfrak{K}_+ \\ \sum_{n=-\infty}^{\infty} (z + n)^k D_{n,n}(K), & K \in \mathfrak{K}_0 \\ \sum_{n=0}^{\infty} (z - n - \frac{1}{2})^k \bar{E}_{n,n}(K), & K \in \mathfrak{K}_- \end{cases}$$

depending on expression parameters. As it is noted, each of them converges in $Hol(\mathbb{C}^2)$.

Consider now the componentwise calculation for r.h.s.. We define matrices for every $k \in \mathbb{Z}$ by

$$(4.17) \quad : (z + \frac{1}{i\hbar} u \circ v)_*^k :_{Kmat} = \begin{cases} \sum_{n=0}^{\infty} (z + n + \frac{1}{2})^k E_{n,n}(K), & K \in \mathfrak{K}_+ \\ \sum_{n=-\infty}^{\infty} (z + n)^k D_{n,n}(K), & K \in \mathfrak{K}_0 \\ \sum_{n=0}^{\infty} (z - n - \frac{1}{2})^k \bar{E}_{n,n}(K), & K \in \mathfrak{K}_- \end{cases}$$

but we often omit the suffix K if the expression parameter is not strictly specified and denote simply by $(z + \frac{1}{i\hbar} u \circ v)_*^k$. The goal of this section is the following theorem:

Theorem 4.3 *If $k = -1$, the r.h.s. of (4.17) defines respectively a holomorphic mapping as follows:*

- (1) *If $K \in \mathfrak{K}_+$, then $\sum_{n=0}^{\infty} (z + n + \frac{1}{2})^{-1} E_{n,n}(K)$ is a holomorphic mapping of $\mathbb{C} \setminus \{-(\mathbb{N} + \frac{1}{2})\}$ into $Hol(\mathbb{C}^2)$,*
- (2) *If $K \in \mathfrak{K}_0$, then $\sum_{n=-\infty}^{\infty} (z + n)^{-1} D_{n,n}(K)$ is a holomorphic mapping of $\mathbb{C} \setminus \mathbb{Z}$ into $Hol(\mathbb{C}^2)$,*
- (3) *If $K \in \mathfrak{K}_-$, then $\sum_{n=0}^{\infty} (z - n - \frac{1}{2})^{-1} \bar{E}_{n,n}(K)$ is a holomorphic mapping of $\mathbb{C} \setminus \{\mathbb{N} + \frac{1}{2}\}$ into $Hol(\mathbb{C}^2)$.*

Now suppose $K \in \mathfrak{K}_+$. Theorem 4.2 gives $: e_*^{s(z + \frac{1}{i\hbar} u \circ v)} :_K = \sum_{k \geq 0} E_{k,k}(K) e^{s(z + k + \frac{1}{2})}$. If $\text{Re } z > -\frac{1}{2}$, then the termwise integration gives

$$\int_{-\infty}^0 : e_*^{s(z + \frac{1}{i\hbar} u \circ v)} :_K ds = \sum_{k=0}^{\infty} \int_{-\infty}^0 E_{k,k}(K) e^{s(z + k + \frac{1}{2})} ds = \sum_{k=0}^{\infty} E_{k,k}(K) (z + k + \frac{1}{2})^{-1}$$

As it is assumed in generic ordered expression, $e_*^{s \frac{1}{i\hbar} u \circ v}$ has no singular point on $s \in \mathbb{R}$, and $e^{-\frac{1}{2}|s|}$ -growth in a generic ordered expression.

To extend this to the domain $\text{Re } z > -n - \frac{1}{2}$, we subtract first divergent terms

$$\sum_{k \geq n} E_{k,k}(K) e^{s(z + k + \frac{1}{2})} = : e_*^{s(z + \frac{1}{i\hbar} u \circ v)} :_K - \sum_{k=0}^{n-1} E_{k,k}(K) e^{s(z + k + \frac{1}{2})},$$

and set on the domain $\text{Re } z > -n - \frac{1}{2}$

$$\sum_{k=0}^{\infty} (z + k + \frac{1}{2})^{-1} E_{k,k}(K) = \sum_{k=0}^{n-1} (z + k + \frac{1}{2})^{-1} E_{k,k}(K) + \int_{-\infty}^0 \left(\sum_{k \geq n} E_{k,k}(K) e^{s(z + k + \frac{1}{2})} \right) ds$$

The r.h.s. converges in $Hol(\mathbb{C}^2)$ in the sense of partial fractions for $\text{Re } z > -n - \frac{1}{2}$

$$\int_{-\infty}^0 \left(\sum_{k \geq n} E_{k,k}(K) e^{s(z + k + \frac{1}{2})} \right) ds = \sum_{k \geq n} E_{k,k}(K) (z + k + \frac{1}{2})^{-1}$$

under the assumption $K \in \mathfrak{K}_+$. Hence we have (1) of Theorem 4.3. A similar proof gives (3) of Theorem 4.3.

For (2) of Theorem 4.3, we show the next one:

Proposition 4.6 *If $K \in \mathfrak{K}_0$, then*

$$D_K^{-1}(z + \frac{1}{i\hbar}u \circ v) = \sum_{n=-\infty}^{\infty} \frac{1}{z+n} D_{n,n}(K)$$

defines a holomorphic mapping from $z \in \mathbb{C} \setminus \mathbb{Z}$ into $Hol(\mathbb{C}^2)$, and this gives an inverse of $z + \frac{1}{i\hbar}u \circ v$ i.e.

$$:(z + \frac{1}{i\hbar}u \circ v):_K *_K D_K^{-1}(z + \frac{1}{i\hbar}u \circ v) = 1, \quad z \in \mathbb{C} \setminus \mathbb{Z}.$$

Proof Note that $D_{n,n}(K)$, $(z+n)D_{n,n}(K)$, $\frac{1}{z+n}D_{n,n}(K) \in Hol(\mathbb{C}^2)$ for every n . For every $z \in \mathbb{C}$, there is an integer $n(z)$ such that $n(z)-1 < \operatorname{Re} z < n(z)+1$. (This may not be unique.)

As $e_*^{t(z + \frac{1}{i\hbar}u \circ v)} = \sum_n e^{t(z+n)} D_{n,n}(K)$ in the topology of $Hol(\mathbb{C}^2)$, we set as follows

$$e_*^{t(z + \frac{1}{i\hbar}u \circ v)} = \sum_{n \leq n(z)-1} e^{t(z+n)} D_{n,n}(K) + \sum_{n(z)-1 < n < n(z)+1} e^{t(z+n)} D_{n,n}(K) + \sum_{n \geq n(z)+1} e^{t(z+n)} D_{n,n}(K).$$

These three terms are members of $Hol(\mathbb{C}^2)$.

Consider inverses of each three term of the r.h.s.. The inverses of the first and the third term can be replaced by using integrals. Hence $D_K^{-1}(z + \frac{1}{i\hbar}u \circ v)$ is written as follows:

$$- \int_0^\infty \sum_{n \leq n(z)-1} e^{t(z+n)} dt D_{n,n}(K) + \sum_{n(z)-1 < n < n(z)+1} (z+n)^{-1} D_{n,n}(K) + \int_{-\infty}^0 \sum_{n \geq n(z)+1} e^{t(z+n)} D_{n,n}(K).$$

By Theorem 4.2, this converges in $Hol(\mathbb{C}^2)$ to give $\sum_{n=-\infty}^\infty \frac{1}{z+n} D_{n,n}(K)$. □

By the definition, we see

Proposition 4.7 $\operatorname{Res}((D_K^{-1}(z + \frac{1}{i\hbar}u \circ v); n) = D_{n,n}(K)$, $n \in \mathbb{Z}$.

Note for (1) and (3). Besides the concrete form of inverses, next two integrals

$$\begin{aligned} (z + \frac{1}{i\hbar}u \circ v)_{*+}^{-1} &= \int_{-\infty}^0 e_*^{s(z + \frac{1}{i\hbar}u \circ v)} ds \quad (\operatorname{Re} z > -\frac{1}{2}), \\ (z + \frac{1}{i\hbar}u \circ v)_{-*}^{-1} &= - \int_0^\infty e_*^{s(z + \frac{1}{i\hbar}u \circ v)} ds \quad (\operatorname{Re} z < \frac{1}{2}). \end{aligned}$$

converges in generic ordered expression on each domain, and these give inverses of $z + \frac{1}{i\hbar}u \circ v$ respectively. In the next section we give analytic continuations of these.

4.2.1 Analytic continuation of inverses

Using the half-inverse v° given by (2.14), we can give the analytic continuation of inverses. First, we see that

$$v^\circ * (z + \frac{1}{i\hbar}u \circ v)_{*+}^{-1} = u * (\frac{1}{2} + \frac{1}{i\hbar}u \circ v)_{*+}^{-1} * (z + \frac{1}{i\hbar}u \circ v)_{*+}^{-1} = u * \frac{1}{z - \frac{1}{2}} \left((\frac{1}{2} + \frac{1}{i\hbar}u \circ v)_{*+}^{-1} - (z + \frac{1}{i\hbar}u \circ v)_{*+}^{-1} \right).$$

Hence $v^\circ * (z + \frac{1}{i\hbar} u \circ v)^{-1}_{*+} * v$ is well-defined, if $z \neq \frac{1}{2}$. More directly we have

$$\begin{aligned} v^\circ * (z + \frac{1}{i\hbar} u \circ v)^{-1}_{*+} &= \left(u * \int_{-\infty}^0 e_*^{t(\frac{1}{i\hbar} u \circ v + \frac{1}{2})} dt \right) * \int_{-\infty}^0 e_*^{s(z + \frac{1}{i\hbar} u \circ v)} ds = u * \int_{-\infty}^0 \int_{-\infty}^0 e_*^{t(\frac{1}{i\hbar} u \circ v + \frac{1}{2})} * e_*^{s(z + \frac{1}{i\hbar} u \circ v)} dt ds \\ &= \int_{-\infty}^0 \int_{-\infty}^0 e^{t\frac{1}{2} + sz} u * e_*^{(t+s)\frac{1}{i\hbar} u \circ v} dt ds \end{aligned}$$

Furthermore, if $(z - 1 + \frac{1}{i\hbar} u \circ v)^{-1}_{*+}$ is already defined, then we continue to compute as follows:

$$= \int_{-\infty}^0 \int_{-\infty}^0 e^{t\frac{1}{2} + sz - (t+s)} e_*^{(t+s)\frac{1}{i\hbar} u \circ v} * u dt ds.$$

Hence, we have the identity whenever both sides are defined:

$$\begin{aligned} (v^\circ * (z + \frac{1}{i\hbar} u \circ v)^{-1}_{*+}) * v &= \int_{-\infty}^0 \int_{-\infty}^0 e^{-t\frac{1}{2} + s(z-1)} e_*^{(t+s)\frac{1}{i\hbar} u \circ v} * (u * v) dt ds \\ &= \int_{-\infty}^0 (u * v) * e_*^{t\frac{1}{i\hbar} u \circ v} dt * \int_{-\infty}^0 e_*^{s(z-1 + \frac{1}{i\hbar} u \circ v)} ds = (1 - \varpi_{00}) * (z - 1 + \frac{1}{i\hbar} u \circ v)^{-1}_{*+}. \end{aligned}$$

Noting that

$$\varpi_{00} * (z - 1 + \frac{1}{i\hbar} u \circ v)^{-1}_{*+} = (z - 1 + \frac{1}{i\hbar} u \circ v)^{-1}_{*+} * \varpi_{00} = (z - \frac{1}{2})^{-1} \varpi_{00},$$

whenever $(z - 1 + \frac{1}{i\hbar} u \circ v)^{-1}_{*+}$ is defined, we have

$$(4.18) \quad (v^\circ * (z + \frac{1}{i\hbar} u \circ v)^{-1}_{*+}) * v + (z - \frac{1}{2})^{-1} \varpi_{00} = (z - 1 + \frac{1}{i\hbar} u \circ v)^{-1}_{*+}.$$

Since $(z - \frac{1}{2})^{-1} \varpi_{00}$ is defined for $z \neq 1/2$, we see that (4.18) gives the formula for analytic continuation. Namely, replacing z by $z+1$, we define the r.h.s. by the l.h.s.

$$(v^\circ * (z + 1 + \frac{1}{i\hbar} u \circ v)^{-1}_{*+}) * v + (z + \frac{1}{2})^{-1} \varpi_{00} = (z + \frac{1}{i\hbar} u \circ v)^{-1}_{*+}, \quad \text{Re } z > -\frac{3}{2}.$$

Repeating this, we have the following formula: For $\text{Re } z > -(n + \frac{1}{2})$,

$$\begin{aligned} (4.19) \quad (z + \frac{1}{i\hbar} u \circ v)^{-1}_{*+} &= \sum_{k=0}^{n-1} (z + k + \frac{1}{2})^{-1} (v^\circ)^k * \varpi_{00} * v^k + (v^\circ)^n * (z + n + \frac{1}{i\hbar} u \circ v)^{-1}_{*+} * v^n, \\ (z - \frac{1}{i\hbar} u \circ v)^{-1}_{*-} &= - \sum_{k=0}^{n-1} (-z + k + \frac{1}{2})^{-1} (u^\bullet)^k * \varpi_{00} * u^k - (u^\bullet)^n * (-z + n - \frac{1}{i\hbar} u \circ v)^{-1}_{*-} * u^n. \end{aligned}$$

By using (2.15) these may be written for z such as $\text{Re } z > -n - \frac{1}{2}$

$$\begin{aligned} (z + \frac{1}{i\hbar} u \circ v)^{-1}_{*+} &= \sum_{k=0}^{n-1} (z + k + \frac{1}{2})^{-1} E_{k,k} + (v^\circ)^n * (z + n + \frac{1}{i\hbar} u \circ v)^{-1}_{*+} * v^n, \\ (z - \frac{1}{i\hbar} u \circ v)^{-1}_{*-} &= \sum_{k=0}^{n-1} (z - k - \frac{1}{2})^{-1} \bar{E}_{k,k} + (u^\bullet)^n * (z - n - \frac{1}{i\hbar} u \circ v)^{-1}_{*-} * u^n. \end{aligned}$$

(See also a comment after Proposition 2.5.)

Note that if $z=z_0$ is fixed then for a sufficiently large n , $(v^\circ)^n * (z+n+\frac{1}{i\hbar}u \circ v)^{-1}_{*+} * v^n$ is holomorphic on a neighborhood of z_0 .

The residue at a singular point z_0 is given as usual by $\frac{1}{2\pi i} \int_{C_{z_0}} (z+\frac{1}{i\hbar}u \circ v)^{-1}_{*\pm} dz$, where C_{z_0} is a small circle with the center at z_0 . The analytic continuation formula (4.19) gives the following:

Theorem 4.4 $\text{Res}((z+\frac{1}{i\hbar}u \circ v)^{-1}_{*+}, -(n+\frac{1}{2}))$ is $\frac{1}{(i\hbar)^n n!} u^n * \overline{\omega}_{00} * v^n$ in generic ordered expressions. This is $E_{n,n}$ by (4.10).

Similarly, $\text{Res}((z-\frac{1}{i\hbar}u \circ v)^{-1}_{*-}, -(n+\frac{1}{2}))$ is $-\frac{1}{(i\hbar)^n n!} v^n * \overline{\omega}_{00} * u^n$ in generic ordered expressions. This is $-\overline{E}_{n,n}$ (cf. (4.11)).

It is remarkable that the singular points depend only on the growth order of $e_*^{t\frac{1}{i\hbar}u \circ v}$ which is independent of the expression parameters.

Since $(z+\frac{1}{i\hbar}u \circ v)^{-1}_{*-} = -(-z-\frac{1}{i\hbar}u \circ v)^{-1}_{*+}$, Theorem 4.4 shows also

Theorem 4.5 In generic ordered expressions, the inverses $(z+\frac{1}{i\hbar}u \circ v)^{-1}_{*+}$, $(z-\frac{1}{i\hbar}u \circ v)^{-1}_{*-}$ extend to $\text{Hol}(\mathbb{C}^2)$ -valued holomorphic functions of z on $\mathbb{C} \setminus \{-(\mathbb{N}+\frac{1}{2})\}$ with simple poles. Namely, for every n

$$:(z+\frac{1}{i\hbar}u \circ v)^{-1}_{*+} \cdot_K - \sum_{k=0}^n (z+k+\frac{1}{2})^{-1} E_{k,k}(K), \quad :(z+\frac{1}{i\hbar}u \circ v)^{-1}_{*-} \cdot_K - \sum_{k=0}^n (z-k-\frac{1}{2})^{-1} \overline{E}_{k,k}(K)$$

are holomorphic on the domain $\text{Re } z > -n-\frac{1}{2}$. However,

$$\sum_{k=0}^{\infty} (z+k+\frac{1}{2})^{-1} E_{k,k}(K), \quad \sum_{k=0}^{\infty} (z-k-\frac{1}{2})^{-1} \overline{E}_{k,k}(K)$$

may not converge in $\text{Hol}(\mathbb{C}^2)$ in the sense of partial fractions. They converge only for $K \in \mathfrak{K}_+$, $K \in \mathfrak{K}_-$ respectively.

The next result may sound strange

Theorem 4.6 If $K \in \mathfrak{K}_0$, then those three elements are inverse of $z+\frac{1}{i\hbar}u \circ v$:

$$:(z+\frac{1}{i\hbar}u \circ v)^{-1}_{*+} \cdot_K, \quad \sum_{n=-\infty}^{\infty} (z+n)^{-1} D_{n,n}(K), \quad :(z+\frac{1}{i\hbar}u \circ v)^{-1}_{*-} \cdot_K$$

They are holomorphic mappings respectively of $\mathbb{C} \setminus \{-(\mathbb{N}+\frac{1}{2})\}$, $\mathbb{C} \setminus \{\mathbb{Z}\}$, $\mathbb{C} \setminus \{\mathbb{N}+\frac{1}{2}\}$ into $\text{Hol}(\mathbb{C}^2)$.

Now, for a fixed $K \in \mathfrak{K}_0$, consider $z \in \mathbb{C}$ where $:(z+\frac{1}{i\hbar}u \circ v) \cdot_K$ fails to be invertible, which may be called the “spectre” of $\frac{1}{i\hbar}u \circ v$ as in the operator theory. Theorem 4.6 shows that $:\frac{1}{i\hbar}u \circ v \cdot_K$ cannot be viewed as a single element from a view point of operator representations, as this has three different kinds of specters

References

- [1] G. Andrews, R. Askey, R. Roy, SPECIAL FUNCTIONS, Encyclopedia Math, Appl. 71, Cambridge, 2000.
- [2] G. S. Agawal, E. Wolf, *Calculus for functions of noncommuting operators and general phase-space method of functions*, Physcal Review D, vol.2, no.10, 1970, 2161-2186.
- [3] F.Bayen, M.Flato, C.Fronsdal, A.Lichnerowicz, D.Sternheimer, *Deformation theory and quantization I, II*, Ann. Phys. 111, (1977), 61-151.
- [4] I.M.Gel'fand, G.E.Shilov, Generalized Functions, 2, Acad. Press, 1968.
- [5] V. Guillemin, S. Sternberg, GEOMETRIC ASYMPTOTICS. A.M.S. Mathematical surveys, 14, 1977.
- [6] H. Omori, INFINITE DIMENSIONAL LIE GROUPS, AMS Translation Monograph 158, 1997.
- [7] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, *Strange phenomena related to ordering problems in quantizations*, Jour. Lie Theorey, Vol 13, no.2, (2003) 491-510.
- [8] H.Omori, Y.Maeda, N.Miyazaki and A.Yoshioka : *Deformation of expressions for elements of algebras*, arxiv.org/abs/1104.1708
- [9] H.Omori, Y.Maeda, N.Miyazaki and A.Yoshioka : *Deformation of expressions for elements of algebras (I)*, arxiv.org/abs/1104.2109
- [10] H.Omori, Y.Maeda, N.Miyazaki and A.Yoshioka : *Deformation of expressions for elements of algebras (II)*, arxiv.org/abs/1105.1218
- [11] H.Omori, Y.Maeda, N.Miyazaki and A.Yoshioka : *Deformation of expressions for elements of algebras (III)*, arxiv.org/abs/1107.2474